Synchronization of Coupled Oscillators is a Game

Huibing Yin, *Member, IEEE*, Prashant G. Mehta, *Member, IEEE*, Sean P. Meyn, *Fellow, IEEE*, and Uday V. Shanbhag, *Member, IEEE*

***Abstract*—The purpose of this paper is to understand phase transition in noncooperative dynamic games with a large number of agents. Applications are found in neuroscience, biology, and economics, as well as traditional engineering applications. The focus of analysis is a variation of the large population linear quadratic Gaussian (LQG) model of Huang *et al.* 2007, comprised here of a controlled N-dimensional stochastic differential equation model, coupled only through a cost function. The states are interpreted as phase angles for a collection of heterogeneous oscillators, and in this way the model may be regarded as an extension of the classical coupled oscillator model of Kuramoto. A deterministic PDE model is proposed, which is shown to approximate the stochastic system as the population size approaches infinity. Key to the analysis of the PDE model is the existence of a particular Nash equilibrium in which the agents 'opt out' of the game, setting their controls to zero, resulting in the 'incoherence' equilibrium. Methods from dynamical systems theory are used in a bifurcation analysis, based on a linearization of the partial differential equation (PDE) model about the incoherence equilibrium. A critical value of the control cost parameter is identified: above this value, the oscillators are incoherent; and below this value (when control is sufficiently cheap) the oscillators synchronize. These conclusions are illustrated with results from numerical experiments.**

***Index Terms*—Mean-field game, Nash equilibrium, nonlinear systems, phase transition, stochastic control, synchronization.**

I. Introduction

The dynamics of a large population of coupled heterogeneous nonlinear systems is of interest in a number of applications, including neuroscience, communication networks, power systems, and economic markets. Game theory provides a powerful set of tools for analysis and design of strategic behavior in controlled multi-agent systems. In economics, for example, game-theoretic techniques provide a foundation for analyzing the behavior of rational agents in markets. In practice, a fundamental problem is that controlled multi-agent systems can exhibit phase transitions with often undesirable outcomes. In economics, an example of this is the so-called *rational irrationality*: "behavior that, on the individual level, is perfectly reasonable but that, when aggregated in the marketplace, produces calamity [3]."

A prototypical example of multi-agent heterogeneous nonlinear system that exhibits phase transition is the coupled oscillator model of Kuramoto [4]. The model comprises of N oscillators, with the ith oscillator's dynamics given by

$$\mathrm{d}\theta_i(t) = \omega_i \mathrm{d}t + \frac{\kappa}{N}\sum_{j=1}^{N} \psi^{\bullet}\left(\theta_j(t), \theta_i(t)\right)\mathrm{d}t + \sigma \mathrm{d}\xi_i(t) \quad (1)$$

where $\theta_i(t)$ is the phase of the ith-oscillator at time t, ω_i is its natural frequency, $\xi_i(t)$ is the standard Wiener process, $\psi^{\bullet}(\theta_j, \theta_i) = \sin(\theta_j - \theta_i)$ models the influence of the jth-oscillator from the population of N oscillators, and κ is the coupling parameter. The frequency ω_i is drawn from a distribution $g(\omega)$ with support on $\Omega := [1-\gamma, 1+\gamma]$. The parameters γ and κ are used to model the heterogeneity and the strength of network coupling, respectively.

The dynamics of the Kuramoto model can be visualized using a bifurcation diagram in the (κ, γ) plane, which in particular illustrates the emergence of a phase transition. The stability boundary $\kappa = \kappa_c(\gamma)$ shown on the left-hand side of Fig. 1 provides an illustration of the phase transition: The oscillators behave incoherently for $\kappa < \kappa_c(\gamma)$, and synchronize for $\kappa > \kappa_c(\gamma)$. That is, the oscillators synchronize if the coupling is sufficiently large. In the former incoherent setting, the oscillators rotate close to their own natural frequency and hence the trajectory $\theta_i(t)$ is approximately independent of the population. In the synchronized setting each oscillator rotates with a common frequency.

The phase transition is important in a number of applications. For example, in thalamocortical circuits in the brain, transition to the synchronized state is associated with diseased brain states such as epilepsy [5].

The objective of this paper is to model and interpret the phase transition from the perspective of noncooperative game theory. We define the game formally:

Consider a set of N oscillators. The model for the ith oscillator is given by

$$\mathrm{d}\theta_i(t) = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma \mathrm{d}\xi_i(t), \qquad \mod 2\pi$$

where $u_i(t)$ is the control input. In neuroscience applications, an oscillator serves as a reduced order model of a single neuron: $\theta_i(t) \in [0, 2\pi]$ is the phase variable and $u_i(t)$ models the effect of stimulus (current) [6]. It is noted that the state space for the ith oscillator is a circle, and denoted as $[0, 2\pi]$.

Suppose the ith oscillator minimizes its own performance objective:

$$\eta_i^{(\mathrm{POP})}(u_i; u_{-i}) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(c(\theta_i; \theta_{-i}) + \frac{1}{2}R u_i^2 \right) \mathrm{d}s \quad (2)$$

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H. Yin and P. G. Mehta are with the Coordinated Science Laboratory and the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, (UIUC), Urbana, IL 61801 USA (e-mail: yin3@illinois.edu; mehtapg@illinois.edu).

S. P. Meyn is with the Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611 USA (e-mail: meyn@ece.ufl.edu).

U. V. Shanbhag is with the Department of Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign, (UIUC), Urbana, IL 61801 USA (e-mail: udaybag@illinois.edu).

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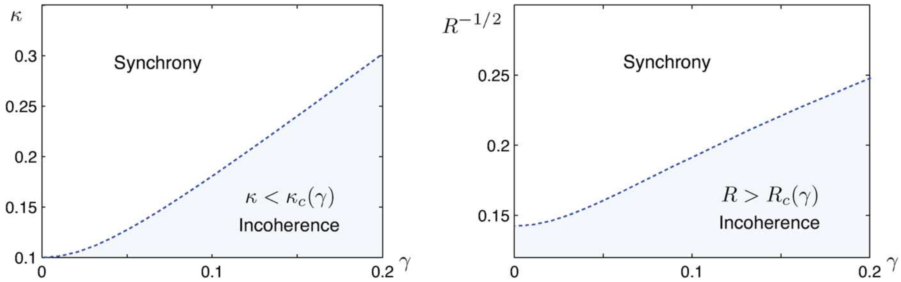

Fig. 1. Bifurcation diagrams. The Kuramoto model (1) with $\sigma^2/2 = 0.05$ (left), and the coupled model considered in this paper with $\sigma^2/2 = 0.05$ (right).

where $\theta_{-i} = (\theta_j)_{j \neq i}$, $c(\cdot)$ is a cost function, $u_{-i} = (u_j)_{j \neq i}$, and R models the control penalty. The form of the function c and the value of R are assumed to be common to the entire population. A *Nash equilibrium* in control policies is given by $\{u_i^*\}_{i=1}^N$ such that u_i^* minimizes $\eta_i^{(\mathrm{POP})}(u_i; u_{-i}^*)$ for $i = 1, \ldots, N$.

In general, establishing the existence and uniqueness of Nash equilibrium for large N is a challenging problem. In this paper, following the *Nash Certainty Equivalence* (NCE) methodology first developed in [2], we investigate a distributed control law wherein the ith oscillator optimizes by using local information consisting of i) its own state (θ_i) and ii) the *mass-influence* of the population. The idea is that in the limit of large population size (as $N \to \infty$), the population affects the ith oscillator in a nearly deterministic fashion. The distributed control law is obtained by considering a problem where each oscillator optimizes with respect to this deterministic (but not a priori known) mass influence.

Three types of analyses are presented in this paper. We first examine the infinite-oscillator limit, and subsequently investigate the implications for the finite-oscillator model:

1) ***The infinite oscillator limit***. A limiting model is constructed consisting of two partial differential equations (PDEs):
   i) A Hamilton–Jacobi–Bellman (HJB) equation (14a) that describes the solution of minimizing (2) under the assumption of a known deterministic mass influence.
   ii) A Fokker–Planck–Kolmogorov (FPK) equation (14b) that describes the evolution of the population density with optimal control input obtained from the solution of i).

   The two PDEs are coupled via the mass influence term (14c). It arises as a (spatially) averaged cost function, where the average is based on the solution of ii). The averaged cost function is used in the HJB equation in i), whose solution defines the distributed control law.
2) ϵ-***Nash equilibrium for finite*** N. Following the methodology outlined in [2], we establish that the distributed control law is an ϵ-Nash equilibrium for the stochastic dynamic game with a finite large number of oscillators $(N < \infty)$. This implies that any unilateral deviation by an individual oscillator can at best improve the performance by a small $(\epsilon = \mathcal{O}(1/\sqrt{N}))$ amount when the population size N is large.

   The final analysis is grounded in the large population limit.
3) ***Transition from incoherence to synchrony***. A bifurcation diagram is obtained in the (R, γ) plane for the infinite limit model. The plot shown on the right in Fig. 1 depicts a phase transition: for $R > R_c$, the oscillators are incoherent, and for $R < R_c$ the oscillators synchronize. That is, the oscillators synchronize when the control is sufficiently cheap.

The bifurcation diagram in Fig. 1 is obtained via spectral analysis of the linearization taken about the incoherent solution. The analysis is used to establish linear asymptotic stability of the incoherence solution for $R > R_c$. For $R < R_c$, the incoherence solution loses stability to a traveling wave solution, interpreted here as the synchrony solution. These solutions are obtained here using a Lyapunov–Schmidt based perturbation method for the homogeneous population, and using a numerical wave form relaxation algorithm for the general heterogeneous case. For the numerical solutions, a brief comparison to the Kuramoto oscillators is also provided.

The overall approach of this paper follows the NCE methodology introduced in the seminal work of Huang, Caines, and Malhamé [2], [7], [8]. In [2], a solution of a cost-coupled LQG game is described for a population comprising of heterogeneous linear agents with Gaussian noise. Each agent seeks to minimize its own infinite-horizon discounted cost. The average cost extension of this problem appears in Li and Zhang [9].

There has also been a parallel development of closely related concepts, referred to as *mean-field games*, beginning with the work of Lasry and Lions [10]. In Gueant [11], a reference case for mean-field game models is discussed: agents have a utility flow that is a function of the population distribution. Numerical approaches to obtain solution of the mean-field game models appears in Achdou and Dolcetta [12].

In the economics literature, mean-field approaches for games with large number of players has a rich history; cf. [13] for an early reference. More recently, the notion of *oblivious equilibrium* was introduced by Weintraub *et al.* [14], [15] as a means of approximating a Markov perfect equilibrium (MPE) for economic models. Related methods to construct approximate solutions to large-scale stochastic games, where state of an individual agent evolves as a discrete-time Markov process, can be found in [16], [17].

The paper is organized as follows. A description of the SDE and PDE models is contained in Section II, and Section III contains analysis of the game for a finite number of oscillators. Bifurcation analysis appears in Section IV, which is illustrated with results from numerical experiments in Section V. Conclusions are contained in Section VI.

II. Mean-Field Approximation

We begin with a description of the coupled oscillator model, associated optimal control problems, and the proposed infinite-limit approximation.

A. Finite Oscillator Model

We consider a population of N oscillators competing in a noncooperative game as defined in the Introduction [see (2)].

The dynamics of the ith oscillator are described by the stochastic differential equation

$$\mathrm{d}\theta_i = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma \mathrm{d}\xi_i, \quad i = 1, \ldots, N, \quad t \geq 0 \quad (3)$$

where $\theta_i(t) \in [0, 2\pi]$ is the phase of the ith oscillator at time t, $u_i(t)$ is the control input, and $\{\xi_i\}$ are mutually independent standard Wiener processes. The standard deviation σ is independent of i. The SDE model requires frequencies ω_i and initial conditions $\{\theta_i(0)\}$, that are chosen independently according to a given distribution:

Assumption (A1): For each i, $\{(\theta_i(0), \omega_i)\}$ is independent and identically distributed (i.i.d.), independent of $\{\xi_i\}$, with common marginal distribution $(\theta_i(0), \omega_i) \sim p(\theta; 0, \omega) g(\omega)$.

The frequency ω_i is a constant independent of time—It is assumed that at time $t = 0$, the N scalars $\{\omega_i\}$ are chosen independently according to a fixed distribution with density g, which is supported on an interval of the form $\Omega = [1 - \gamma, 1 + \gamma]$ where $\gamma < 1$ is assumed to be a small constant. For a *homogeneous population* $\gamma = 0$ and $g(\omega) = \delta(\omega - 1)$, the Dirac delta function at $\omega = 1$. ■

In the numerical examples described in Section V, the density is taken to be uniform, namely $g(\omega) = (2\gamma)^{-1}$ for $|\omega - 1| \leq \gamma$.

We seek a control solution that is decentralized and of the following form: For each i and t, the input $u_i(t)$ depends only on $\{\theta_i(s) : s \leq t\}$, and perhaps some aggregate information, such as the mean value of $\{\theta_j(t)\}_{j=1}^N$. This amounts to a dynamic game, whose exact solution is infeasible for large N.

Instead we construct an approximation of the form described in [2] and [18]. This approximation is based on an infinite-population limit similar to those introduced in this prior work and others (e.g., [19]). The approximation is based on the following sequence of steps:

1) We construct a density function p that is intended to approximate the probability density function for the individual oscillators in steady-state. For any i and any $t > 0$, the density $p(\cdot, t; \omega_i)$ is intended to approximate the probability density of the random variable $\theta_i(t)$, evolving according to the stochastic differential equation (3).
2) We obtain an approximation for the cost function c. Motivated by the separable nature of the control used in the coupled oscillator models [e.g., (1)], we make the following assumption:

Assumption (A2):

The cost function c appearing in (2) is separable, as shown as follows:

$$c(\theta_i; \theta_{-i}) = \frac{1}{N} \sum_{j \neq i} c^\bullet(\theta_i, \theta_j) \quad (4)$$

with $c^\bullet$ a non-negative integrable function on $[0, 2\pi]^2$. ■
If N is large, the sum in (4) is expected to be nearly deterministic when the frequencies $\{\omega_i\}$ are independently sampled according to the density g. The law of large numbers (LLN) suggests the approximation of $c(\vartheta; \theta_{-i}(t))$ by

$$\bar{c}(\vartheta, t) := \int_{\omega \in \Omega} \int_{\theta=0}^{2\pi} c^\bullet(\vartheta, \theta) p(\theta, t; \omega) g(\omega) \mathrm{d}\theta \mathrm{d}\omega. \quad (5)$$

3) For the scalar model (3) with cost $\bar{c}(\vartheta, t)$ depending only on $\vartheta = \theta_i$, the game reduces to independent optimal control problems. The individual agents are oblivious to the state of the entire system and make their control decisions based only on local state variables.
We show in this paper that this approximation is justified in the limit of large population size.

In the following subsection we develop the "oblivious" solution described in 3). We then turn to the PDE approximation in 1) that defines the approximate cost (5) in 2).

B. Optimal Control of a Single Oscillator

Suppose that a cost function is given for the single-oscillator model: it is possibly time-dependent, and a continuous function of its arguments, of the form

$$\bar{c}(\theta_i, t) + \frac{1}{2} R u_i^2.$$

The average cost is defined as the limit supremum

$$\eta_i(u_i; \bar{c}) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left(\bar{c}(\theta_i(s), s) + \frac{1}{2} R u_i^2(s) \right) \mathrm{d}s. \quad (6)$$

The goal is to minimize η_i over all *admissible* controls. In the context here this means that $u_i(t)$ is adapted to the filtration $\mathcal{X}_t^i := \{\theta_i(s) : s \leq t\}$. We let η_i^* denote the minimal cost.

A *Markov policy* is simply state-feedback, of the form $u_i(t) = \varphi_i(\theta_i(t), t)$. If the feedback law is C^1, then the evolution of the density $p(\theta, t; \omega_i)$ is defined by the Fokker–Planck–Kolmogorov (FPK) equation

$$\partial_t p + \partial_\theta ((\omega + \varphi_i(\theta, t)) p) = \frac{\sigma^2}{2} \partial_{\theta\theta}^2 p \quad (7)$$

where ∂_t and ∂_θ denote the partial derivative with respect to t and θ, respectively, and $\partial_{\theta\theta}^2$ denotes the second derivative with respect to θ.

To characterize the optimal input as C^1 state feedback we turn to the associated average-cost optimality equations (or HJB equations) given by

$$\min_{u_i} \left\{ \bar{c}(\theta, t) + \frac{1}{2} R u_i^2 + \mathcal{D}_{u_i} h_i(\theta, t) \right\} = \eta_i^*. \quad (8)$$

The function $h_i(\theta, t)$ is called the relative value function, η_i^* is the minimal average cost defined above, and $\mathcal{D}_u$ denotes the controlled generator, defined for C^2 functions g via

$$\mathcal{D}_u g = \partial_t g + (\omega_i + u) \partial_\theta g + \frac{\sigma^2}{2} \partial_{\theta\theta}^2 g.$$

Provided a C^2 solution to (8) exists, the minimizer in this equation defines a state feedback control law that is optimal. Because the cost is quadratic in u_i, and the dynamics linear in u_i, the optimal feedback law is expressed as the C^1 function of θ_i

$$\varphi_i(\theta, t) := -\frac{1}{R} \partial_\theta h_i(\theta_i, t). \quad (9)$$

Substituting $u_i^*(t)$ into (8) gives the nonlinear PDE

$$\partial_t h_i + \omega \partial_\theta h_i = \frac{1}{2R} (\partial_\theta h_i)^2 - \bar{c}(\theta, t) + \eta_i^* - \frac{\sigma^2}{2} \partial_{\theta\theta}^2 h_i. \quad (10)$$

Under Assumption (A3), the optimal control (9) is realized as continuous and bounded state feedback.

Assumption (A3): There is a bounded, C^2 solution to (10) whose first derivatives are *uniformly bounded*:

$$\sup_{t,\theta} \{|h_i(\theta,t)| + |\partial_t h_i(\theta,t)| + |\partial_\theta h_i(\theta,t)|\} < \infty. \qquad (11)$$

■

The next result establishes optimality of the Markov policy (9), even in the general time-dependent setting described here. The proof appears in Appendix VII-A.

Proposition 2.1: Consider the single oscillator optimal control problem (6) for the ith-oscillator. Suppose that Assumption (A3) holds. Then, the Markov policy (9) is average cost optimal, with average cost η_i^*, independent of the initial condition $\theta_i(0)$.■

Justification of Assumption A3 is beyond the scope of this paper, but we can give conditions under which a slight relaxation is valid. Suppose that the cost function is periodic, with period $0 \le \tau < \infty$ (if $\tau = 0$, then $\bar{c}$ is independent of time). Denote the space–time process by $\Phi(t) = (\theta_i(t), [t]_\tau)$, where the second variable is simply the time variable, modulo the period τ. The space–time process may be viewed as a controlled Markov process on the product space $[0,2\pi]\times[0,\tau]$, so that cost is only a function of this state and the control u. Following standard arguments [20], it follows that an optimal policy is defined as a *stationary* Markov policy. That is, $u_i(t) = \varphi_i(\theta_i(t), [t]_\tau)$. For a continuous feedback law of this form the controlled diffusion Φ is *hypoelliptic*, for which there is a rich ergodic theory (see Prop. 3.1 below). In particular, because of the compact state space, the average cost (6) exists as a limit and is independent of the initial state [21]. Moreover, for each i there exists a solution to Poisson's equation for the optimal policy [22], and this solves (8) with $\mathcal{D}$ interpreted as the *extended generator* [22], rather than a differential operator.

In contrast to the time-average problem (6) considered here, the existence theory for the discounted cost problem is relatively easier and appears in [2] for the linear quadratic case.

C. PDE Model

We now provide a complete description of the PDE model that is intended to approximate the stochastic model for large N. This model is based on the cost function $\bar{c}(\theta,t)$ introduced in the preceding section. A relative value function $h(\theta,t;\omega)$ and a density $p(\theta,t;\omega)$ for the infinite population model are defined by differential equations identical to those considered for the single oscillator model.

The relative value function h is defined as the solution to an HJB equation, exactly as in (8):

$$\partial_t h + \omega\partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h.$$

The associated feedback control law is then defined as in (9) by

$$\varphi(\theta,t;\omega) := -\frac{1}{R}\partial_\theta h(\theta,t;\omega). \qquad (12)$$

Given the feedback control law (12), the FPK equation that defines the evolution of the normalized density is given by the analog of (7)

$$\partial_t p + \omega\partial_\theta p = \frac{1}{R}\partial_\theta\left[p(\partial_\theta h)\right] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p.$$

The average cost is then defined as the function of ω

$$\eta^*(\omega) = \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^{2\pi}\left[\bar{c}(\theta,t) + \frac{1}{2R}(\partial_\theta h)^2\right]p(\theta,t;\omega)\mathrm{d}\theta\mathrm{d}t. \qquad (13)$$

The only difference thus far is notational: $h_i(\theta,t)$ is the value function for $N=1$ with a single frequency ω_i, and $h(\theta,t;\omega)$ is the value function for a continuum of oscillators, distinguished by their natural frequency ω. Such is the case because we have *assumed* $\bar{c}(\theta,t)$ is a known deterministic function that is furthermore consistent across the population.

All that remains is to specify $\bar{c}(\theta,t)$ in a self-consistent manner. The consistency enforced here is inspired by the approximation given in (5). The two PDEs are coupled through this integral that defines the relationship between the cost $\bar{c}$ and the density p:

$$\bar{c}(\vartheta,t) = \int_\Omega\int_0^{2\pi} c^\bullet(\vartheta,\theta)p(\theta,t;\omega)g(\omega)\mathrm{d}\theta\mathrm{d}\omega.$$

In summary, the PDE model is given by

$$\partial_t h + \omega\partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \qquad (14a)$$

$$\partial_t p + \omega\partial_\theta p = \frac{1}{R}\partial_\theta\left[p(\partial_\theta h)\right] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \qquad (14b)$$

$$\bar{c}(\vartheta,t) = \int_\Omega\int_0^{2\pi} c^\bullet(\vartheta,\theta)p(\theta,t;\omega)g(\omega)\mathrm{d}\theta\mathrm{d}\omega. \qquad (14c)$$

In the remainder of this paper, we investigate solutions $\{\mathfrak{p}(\theta,t;\omega), \mathfrak{h}(\theta,t,\omega)\}$ of this coupled PDE. Under the assumption of periodicity, we restrict to two cases: the *equilibrium* solution in which the cost function, the relative value function, and the density are independent of time, or the *periodic* case in which $\mathfrak{p}(\theta,t;\omega)$, $\mathfrak{h}(\theta,t;\omega)$, and $\bar{c}(\theta,t)$ are periodic in time, with period $\tau > 0$. The equilibrium and periodic solutions are considered for the following reasons:

1) These solutions define approximate Nash equilibrium of the game with a finite large number of oscillators. This is discussed in Section III.
2) For certain values of the parameter R, these solutions represent the steady-state solutions of the PDE model (see Sections IV, V).
3) These solutions potentially represent the incoherence and synchrony solutions described in the coupled oscillators literature [19], [23] (see Figs. 1 and 5).

D. Incoherence

The system of (14a)–(14c) may have multiple solutions. Suppose that the cost $c^\bullet$ introduced in (4) is of the form $c^\bullet(\vartheta,\theta) = c^\bullet(\vartheta-\theta)$. In this case we single out the *incoherence* solution defined by

$$\mathfrak{h}(\theta,t;\omega) = \mathfrak{h}_0(\theta) := 0, \qquad \mathfrak{p}(\theta;t,\omega) = \mathfrak{p}_0(\theta) := \frac{1}{2\pi}.$$

The nomenclature "incoherence solution" is inspired from the coupled (Kuramoto) oscillators literature, where it is used to describe the solution with angles $\{\theta_i(t)\}$ (for the population) uniformly distributed on the circle $[0,2\pi]$ [19].

The control law (12) sets $u(t) \equiv 0$. The cost $\bar{c}$ defined in (5) is constant in this solution.

Consider the special case

$$c^{\bullet}(\vartheta,\theta) = \frac{1}{2}\sin^2\left(\frac{\vartheta-\theta}{2}\right).$$

For the incoherence solution

$$\bar{c}(\vartheta,t) = \frac{1}{2}\frac{1}{2\pi}\int_{\Omega}\int_{0}^{2\pi}\sin^2\left(\frac{\vartheta-\theta}{2}\right)g(\omega)\mathrm{d}\theta\mathrm{d}\omega = \frac{1}{4}$$

which coincides with the average cost $\eta^*(\omega) = \eta_0 := \bar{c}$ for all $\omega \in \Omega$. This value is approximately consistent with the finite-N model. When each control is set to zero we obtain $\mathrm{d}\theta_i(t) = \omega_i \mathrm{d}t + \sigma\mathrm{d}\xi_i(t)$ for each i, which results in average cost independent of i,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T c\left(\theta_i(t);\theta_{-i}(t)\right)\mathrm{d}t = \frac{N-1}{N}\eta_0.$$

We return to this example in the bifurcation analysis of Section IV. There is a tradeoff between reducing the cost associated with $\theta_i \neq \theta_j$, and reducing the cost of control. These competing costs suggest that a qualitative change in optimal control may arise when the parameter R varies from ∞ to 0.

Remark 1: The incoherence solution is a solution of the coupled PDE model (14a)–(14c) obtained here without an explicit requirement of a particular initial condition. One may wonder whether it represents also a steady-state for a certain initial value problem? This is discussed in Sections IV-A–IV-C with the aid of a linear analysis. ∎

III. ϵ-Nash Equilibrium

In this section, we assume that we have a solution $(\mathfrak{h}(\theta,t;\omega),\mathfrak{p}(\theta,t;\omega))$ of the PDE model described in Section II-C. We show that in the stochastic model (3) with $N<\infty$, the resulting control solution φ given in (12) defines an *almost sure* ϵ-Nash equilibrium, with $\epsilon\to 0$ as $N\to\infty$. The concept of almost sure ϵ-Nash equilibrium has been proposed in [9].

A. Oblivious Control

Suppose that $N<\infty$, and that each oscillator uses the feedback control law (12) as follows:

$$u_j^o(t) = \varphi^o\left(\theta_j(t),t;\omega_j\right) := -\frac{1}{R}\partial_\theta\mathfrak{h}\left(\theta_j(t),t;\omega_j\right). \quad (15)$$

We refer to (15) as the *oblivious control* [18].

Here we also recall that the initial conditions are chosen independently: $\{(\theta_i(0),\omega_i)\}$ is i.i.d., independent of $\{\xi_i\}$, with common marginal distribution $(\theta_i(0),\omega_i) \sim p(\theta;0,\omega)g(\omega)$ [see Assumption (A1)].

For a periodic solution, the space-time process $\Phi(t) = (\theta_j(t),[t]_\tau)$ with control (15) is a hypoelliptic diffusion. Theorem 3.2 of [21] implies that the process is ergodic in a strong sense:

Proposition 3.1: Suppose that the control law (15) is periodic, with period τ. For each value of $r\geq 0$ the skeleton chain $\{\theta_j(k\tau+r) : k\geq 0\}$ satisfies Doeblin's condition, and hence possesses a unique invariant measure μ_r with density $\mathfrak{p}(\cdot,k\tau+r,\cdot)$, for any k:

$$\mu_r(A) = \int_A \mathfrak{p}(\theta,k\tau+r;\omega_j)\mathrm{d}\theta,\ \textit{for measurable}\ A\subset[0,2\pi]$$

Moreover, the Markov process Φ satisfies the following ergodic theorems:

1) The skeleton chain is uniformly ergodic: there exists $b<\infty$ and $\delta>0$ such that for any measurable set $A\subset[0,2\pi]$, each initial $\theta_j(0)$, and each $r\geq 0$:

$$|\mathsf{P}\{\theta_j(k\tau+r)\in A\} - \mu_r(A)| \leq be^{-\delta(k\tau+r)}.$$

2) The continuous-time Markov process Φ is positive Harris recurrent with unique invariant measure μ, defined by the time-average of those for the skeleton chains:

$$\mu(A\times B) = \frac{1}{\tau}\int_0^\tau \mu_r(A)\mathbb{1}\{r\in B\}dr,\ \ A\subset[0,2\pi], B\subset[0,\tau]. \quad (16)$$

3) The LLN holds for each bounded and measurable function $f:[0,2\pi]\times[0,\tau]\to\mathbb{R}$:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\Phi(t)\right)dt = \int f(x)\mu(dx).$$

∎

B. Optimal Control of a Single Oscillator Revisited

Suppose each oscillator except for the ith oscillator applies the oblivious feedback law (15). In this case, the ith oscillator faces an ordinary stochastic control problem:

$$\eta_i^{(\mathrm{POP})}\left(u_i;u_{-i}^o\right) = \limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\frac{1}{N}\sum_{j\neq i}c^{\bullet}\left(\theta_i(s),\theta_j^o(s)\right) + \frac{1}{2}Ru_i^2\right)\mathrm{d}s \quad (17)$$

where the notation $\theta_j^o(t)$ is used to denote angle of the jth oscillator at time t with oblivious control $u_j^o(t)$. The goal is to minimize $\eta_i^{(\mathrm{POP})}$ over all *admissible* controls. In the context here this means that $u_i(t)$ is adapted to the filtration $\mathcal{X}_t := \{(\theta_i(s),\theta_{-i}(s)) : s\leq t\}$.

The state process (θ_i,θ_{-i},t) is $(N+1)$-dimensional, and whose controlled generator $\mathcal{D}_u^o$ is subject to the fixed policies used by the other oscillators. It is defined for C^2 functions g via

$$\mathcal{D}_u^o g = \partial_t g + (\omega_i+u)\partial_{\theta_i}g + \frac{1}{2}\sigma^2\partial^2_{\theta_i\theta_i}g + \sum_{j\neq i}\left(\left(\omega_j+\varphi^o(\theta_j,t;\omega_j)\right)\partial_{\theta_j}g + \frac{1}{2}\sigma^2\partial^2_{\theta_j\theta_j}g\right).$$

We state without proof the following standard proposition for the solution of this problem (see, e.g., [20]):

Proposition 3.2: Suppose that $i\in\{1,\dots,N\}$ is fixed, and the oblivious policy (15) is fixed for each $j\neq i$. Then:

1) The optimal control problem (17) for the ith oscillator is characterized by the controlled diffusion with state space $\mathsf{X} := [0,2\pi]^N\times[0,\tau]$.

2) The average-cost optimality equation is given by

$$\min_u \left\{ \frac{1}{N} \sum_{j \neq i} c^\bullet(\theta_i, \theta_j) + \frac{1}{2} R u_i^2 + \mathcal{D}_u^o h^*(\theta_i, \theta_{-i}, t) \right\} = \eta_i^* \tag{18}$$

where $h^*(\theta_i, \theta_{-i}, t)$ is the relative value function.

3) Suppose that there exists a C^2 solution to (18), with bounded derivatives as in (11) of Assumption (A3). Then, the minimal average cost over all admissible controls $\{u_i(t)\}$ coincides with the minimum over periodic state feedback policies of the form $u_i(t) = \varphi(\theta_i(t), \theta_{-i}(t), [t]_\tau)$, where the function $\varphi : \mathsf{X} \to \mathbb{R}$ is continuous. ■

We henceforth restrict to feedback laws of the periodic state-feedback form as in 3). Under this condition we can extend Prop. 3.1 to the $(N+1)$-dimensional state process:

Proposition 3.3: For fixed i, suppose that each oscillator except for oscillator i applies the oblivious feedback law (15), and that oscillator i applies a periodic state feedback solution of the form $u_i(t) = \varphi^N(\theta_i(t), \theta_{-i}(t), [t]_\tau)$, where the function $\varphi^N : \mathsf{X} \to \mathbb{R}$ is continuous. Then, for each value of $r \geq 0$, the $N+1$-dimensional skeleton chain $\{\theta(k\tau + r) : k \geq 0\}$ satisfies Doeblin's condition, and is uniformly ergodic. ■

C. Almost Sure ϵ-Nash Property of Oblivious Control

The goal is to show that the oblivious control law (15) is approximately optimal for the ith oscillator provided all other oscillators also use the oblivious control, and N is large.

The key is that for large N, the finite sum in (17) can be approximated by the deterministic function $\bar{c}(\theta_i, t)$ that is defined using the integral (14c) in the PDE limit. The nature of approximation is made precise in the following Proposition 3.4 whose proof, given in Appendix VII-B, is based on the LLN.

Proposition 3.4: Suppose that $i \in \{1, \ldots, N\}$ is fixed, and the oblivious policy (15) is fixed for each $j \neq i$. Suppose furthermore that Assumption (A1) holds, and that the ith oscillator uses a time-periodic feedback control of the form $u_i(t) = \varphi^N(\theta_i(t), \theta_{-i}(t), [t]_\tau)$ where the function $\varphi^N : \mathsf{X} \to \mathbb{R}$ is continuous. Then, there is a sequence of random variables $\{\epsilon_N\}$ such that,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\frac{1}{N} \sum_{j \neq i} c^\bullet\left(\theta_i(s), \theta_j(s)\right) - \bar{c}\left(\theta_i(s), s\right) \right) \mathrm{d}s = \varepsilon_N$$

where $\epsilon_N \to 0$ as $N \to \infty$ a.s., and also in mean square at rate N^{-1}:

$$\limsup_{N\to\infty} N \mathsf{E}\left[\varepsilon_N^2\right] < \infty. \tag{19}$$

■

We can now establish the main result of this section. Recall first that the oblivious control law is designed to be optimal with respect to the deterministic function $\bar{c}(\theta_i, t)$, i.e.,

$$\eta_i\left(u_i^o; \bar{c}\right) \leq \eta(u_i; \bar{c}). \tag{20}$$

The proof of the following theorem appears in Appendix VII-C (see also [2]).

Theorem 3.5: Suppose that Assumption (A1) holds. For large N, the oblivious control $\{u_i^o\}$ is an ϵ-Nash equilibrium for (2): For any admissible control u_i:

$$\eta_i^{(\mathrm{POP})}\left(u_i^o; u_{-i}^o\right) \leq \eta_i^{(\mathrm{POP})}\left(u_i; u_{-i}^o\right) + \varepsilon_N$$

where $\epsilon_N \to 0$ as $N \to \infty$ a.s., and in mean square with rate N^{-1} as in (19). ■

IV. Bifurcation Analysis of PDEs

In the remainder of the paper we present a finer analysis of the coupled equations (14a)–(14c). The following assumption is imposed on the cost:

Assumption (A4): The cost function $c^\bullet$ introduced in (4) is assumed to be an integrable function that is:

1) spatially invariant, i.e., $c^\bullet(\vartheta, \theta) = c^\bullet(\vartheta - \theta)$;
2) non-negative, i.e., $c^\bullet(\theta) \geq 0$;
3) even, i.e., $c^\bullet(\theta) = c^\bullet(-\theta)$. ■

We write the Fourier series of the cost function as

$$c^\bullet(\theta) = C_0^\bullet + \sum_{k=1}^{\infty} C_k^\bullet \cos(k\theta). \tag{21}$$

For the numerical example, we consider $c^\bullet(\vartheta - \theta) = (1/2)\sin^2((\vartheta - \theta)/2))$. In this case, $C_0^\bullet = 1/4$, $C_1^\bullet = -(1/4)$, and $C_k^\bullet = 0$ for $k = 2, 3, \ldots$

Our main goal in this and the following section is to establish a transition from incoherence to synchrony as the control penalty parameter R is decreased beyond a critical value. Before presenting the details of the analysis, we first provide a roadmap of what is to follow:

1) Solutions to the (14a)–(14c) are investigated using the method of bifurcation theory; the parameter R is used as the bifurcation parameter.
2) We single out one solution obtained in Section II-D: The incoherence solution. We denote this solution by $\mathfrak{z}_0(\theta) := (\mathfrak{h}_0(\theta), \mathfrak{p}_0(\theta))^{\mathrm{T}}$. The local stability of the incoherence solution is investigated via analysis of a linearization about $\mathfrak{z}_0$ (Section IV-A). The spectral analysis of the linearization is also used to obtain the bifurcation point as a critical value of $R = R_c(\gamma)$ where the incoherence solution loses stability (Sections IV-B, IV-C).
3) Although the linear analysis is described for the general heterogeneous case, a rigorous bifurcation result for the nonlinear problem is proved only for the homogeneous case (where $g(\omega) = \delta(\omega - 1)$). In the latter case, we show the existence of a small amplitude traveling wave solution via the method of Lyapunov–Schmidt (Section IV-D). A perturbation formula is given for this special case.
4) For the heterogeneous case, the solution of the PDE for a specific cost function $c^\bullet(\vartheta - \theta) = (1/2)\sin^2((\vartheta - \theta)/2)$ is obtained numerically by using an algorithm which can be found in [1]. Numerical results described in Section V show that the incoherent solution is a limiting fixed-point of the algorithm when $R > R_c$. Below the critical value of R, the incoherent solution is no longer stable. The numerical algorithm yields a periodic traveling wave solution that is interpreted as the synchrony solution.

A. Linear PDEs

The linearization of the (14a)–(14c) is taken about the equilibrium incoherence solution $\mathfrak{z}_0 = (\mathfrak{h}_0, \mathfrak{p}_0)$. A perturbation of this solution is denoted $\mathfrak{z}_0 + \tilde{z} = (\mathfrak{h}_0, \mathfrak{p}_0) + (\tilde{h}, \tilde{p})$, where $\tilde{z}(\theta, t; \omega) = \tilde{z}(\theta + 2\pi, t, \omega)$ for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$. Since $p = \mathfrak{p}_0 + \tilde{p}$ is a probability density, the perturbation satisfies the normalization condition $\int_0^{2\pi} \tilde{p}(\theta, t; \omega)\mathrm{d}\theta = 0$ for all t, ω. Since the relative value function is only defined to a constant, we also impose a similar normalization condition for h: $\int_0^{2\pi} \tilde{h}(\theta, t; \omega)\mathrm{d}\theta = 0$ for all t, ω.

When $\tilde{z}$ is small, its evolution is approximated by the linear equation

$$\frac{\partial}{\partial t}\tilde{z}(\theta,t;\omega) = \mathcal{L}_R\tilde{z}(\theta,t;\omega) \tag{22}$$

where

$$\mathcal{L}_R\tilde{z}(\theta,t;\omega) := \begin{pmatrix} -\omega\partial_\theta\tilde{h} - \frac{\sigma^2}{2}\partial^2_{\theta\theta}\tilde{h} \\ -\omega\partial_\theta\tilde{p} + \frac{1}{2\pi R}\partial^2_{\theta\theta}\tilde{h} + \frac{\sigma^2}{2}\partial^2_{\theta\theta}\tilde{p} \end{pmatrix} - \begin{pmatrix} \tilde{c}(\theta,t) \\ 0 \end{pmatrix}$$

and

$$\tilde{c}(\theta,t) = \int_\Omega \int_0^{2\pi} c^\bullet(\theta,\vartheta)\tilde{p}(\vartheta,t;\omega)g(\omega)\mathrm{d}\vartheta\mathrm{d}\omega.$$

The local analysis entails well-posedness (existence, uniqueness) and stability with respect to an infinitesimal initial perturbation of the population density

$$\tilde{p}(\theta,0;\omega) = q(\theta,\omega) \tag{23}$$

where $\int_0^{2\pi} q(\theta,\omega)\mathrm{d}\theta = 0$.

The analysis requires the introduction of a Hilbert space, taken here to be $\mathbf{L}^2(\mathbb{R}^+,\mathbf{H}\times\mathbf{H})$, where $\mathbf{H}$ is a subspace of $\mathbf{L}^2([0,2\pi]\times\Omega)$. The space $\mathbf{L}^2([0,2\pi]\times\Omega)$ is defined with respect to the measure $g(\omega)\mathrm{d}\omega\mathrm{d}\theta$ on the product space $[0,2\pi]\times\Omega$. For any complex-valued function $v(\theta,\omega)$ on $[0,2\pi]\times\Omega$ we denote

$$\|v\|^2_{\mathbf{H}} := \int_0^{2\pi}\int_\Omega |v(\theta,\omega)|^2 g(\omega)\mathrm{d}\omega\mathrm{d}\theta.$$

The Hilbert space $\mathbf{H}$ is defined to be the set of functions for which the integral is finite, and $\int_0^{2\pi} v(\theta,\omega)\mathrm{d}\theta = 0$ for all $\omega\in\Omega$. We denote $\mathbf{H}^2 := \mathbf{H}\times\mathbf{H}$.

We refer to the problem (22), (23) as the linear *initial value problem*. The problem is *well-posed* if a unique solution $(\tilde{h},\tilde{p})(\theta,t;\omega)$ exists in $\mathbf{L}^2(\mathbb{R}^+,\mathbf{H}^2)$ for any initial perturbation $\tilde{p}(\theta,0,\omega) = q(\theta,\omega)\in\mathbf{H}$. Along with well-posedness, we are interested in local stability of the incoherence solution:

Definition 1: Consider the incoherence solution $\mathfrak{z}_0 = (\mathfrak{h}_0,\mathfrak{p}_0)$ of the coupled nonlinear PDE (14a)–(14c). The incoherence solution $\mathfrak{z}_0 = (\mathfrak{h}_0,\mathfrak{p}_0)$ is *linearly asymptotically stable* if a solution $\tilde{p}(\theta,t;\omega)$ of the linear initial value problem (22), (23) with initial perturbation $\tilde{p}(\theta,0;\omega) = q(\theta,\omega)\in\mathbf{H}$ exists in $\mathbf{L}^2(\mathbb{R}^+,\mathbf{H}^2)$, and satisfies $\|\tilde{p}(\theta,t;\omega)\|_{\mathbf{H}}\to 0$ as $t\to\infty$. ■

For the stability analysis of the linear initial value problem (22), (23), it is useful to first deduce the spectra. Since the functions $\tilde{p},\tilde{h},\tilde{c}$ are 2π-periodic, Fourier coordinates are used to obtain a simpler diagonal representation of the linear operator $\mathcal{L}_R$.

Key to the representation is the Fourier series expansion with respect to θ:

$$\begin{aligned}\tilde{h}(\theta,t;\omega) &= \sum_{k=1}^{+\infty} H_k(t,\omega)e^{ik\theta} + \text{c.c}, \\ \tilde{p}(\theta,t;\omega) &= \sum_{k=1}^{+\infty} P_k(t,\omega)e^{ik\theta} + \text{c.c}\end{aligned} \tag{24}$$

where c.c denotes the complex conjugate. We also require expansions of the initial condition $q(\theta,\omega) = \sum_{k=1}^{+\infty} Q_k(\omega)e^{ik\theta} +$ c.c, and the perturbation of the cost function

$$\tilde{c}(\theta,t) = \pi\sum_{k=1}^{\infty} C_k^\bullet \int_\Omega P_k(t,\omega)g(\omega)\mathrm{d}\omega e^{ik\theta} + \text{c.c.} \tag{25}$$

Using (24), (25) yields a diagonal decomposition of the linear operator

$$\mathcal{L}_R = \bigoplus_k \mathcal{L}_R^{(k)}$$

where each $\mathcal{L}_R^{(k)}$ acts on the pair $(H_k,P_k)^{\mathrm{T}}$. The individual operators have the explicit form

$$\mathcal{L}_R^{(k)} := \begin{pmatrix} \frac{\sigma^2}{2}k^2 - k\omega i & -\pi C_k^\bullet \int_\Omega (\cdot)g(\omega)\mathrm{d}\omega \\ -\frac{k^2}{2\pi R} & -\frac{\sigma^2}{2}k^2 - k\omega i \end{pmatrix} \tag{26}$$

and $\mathcal{L}_R^{(-k)} = \overline{\mathcal{L}_R^{(k)}}$, $\forall k\geq 1$.

The Fourier coordinate functions H_k, P_k do not depend upon coordinate θ. For such functions, we introduce the subspace $\mathbf{H}_\varpi\subset\mathbf{H}$: It is defined to be the set of complex-valued function $v(\omega)$ on Ω such that the norm

$$\|v\|^2_{\mathbf{H}_\varpi} := \int_\Omega |v(\omega)|^2 g(\omega)\mathrm{d}\omega$$

is finite.

B. Spectrum

We say that $\lambda\in\mathbb{C}$ is in the spectrum of $\mathcal{L}_R$ if the inverse $[I\lambda - \mathcal{L}_R]^{-1}$ does not exist as a bounded linear operator on $\mathbf{H}^2$. The associated eigenvector problem is given by

$$\lambda z = \mathcal{L}_R z.$$

The spectrum of $\mathcal{L}_R$ is given by the union of spectrum of $\mathcal{L}_R^{(k)}$, $k = \pm 1,\pm 2,\ldots$. In general, the spectrum includes both continuous and discrete parts. In Appendix VII-D we establish the following characterization:

Theorem 4.1: For the linear operator $\mathcal{L}_R:\mathbf{H}^2\to\mathbf{H}^2$,

1) The continuous spectrum equals the union of sets $\{S_c^{(k)}\}_{k\neq 0}$, where

$$S_c^{(k)} := \left\{\lambda\in\mathbb{C}\ \Big|\ \lambda = \frac{\sigma^2}{2}k^2 - k\omega i \text{ for all } \omega\in\Omega\right\}.$$

2) The discrete spectrum equals the union of sets $\{S_d^{(k)}\}_{k\neq 0}$.

We have the following two cases:

1) If $C_{|k|}^\bullet = 0$, the set $S_d^{(k)}$ is empty;
2) if $C_{|k|}^\bullet \neq 0$:

$$S_d^{(k)} := \left\{\lambda\in\mathbb{C}\ \Big|\ \frac{k^2 C_{|k|}^\bullet}{2R} \times \int_\Omega \frac{g(\omega)}{\left(\lambda - \frac{\sigma^2}{2}k^2 + k\omega i\right)\left(\lambda + \frac{\sigma^2}{2}k^2 + k\omega i\right)}\mathrm{d}\omega - 1 = 0\right\}.$$

■

The points in $S_c^{(k)}$ are in one-one correspondence with the frequencies in the support of the distribution $g(\omega)$. That is, for

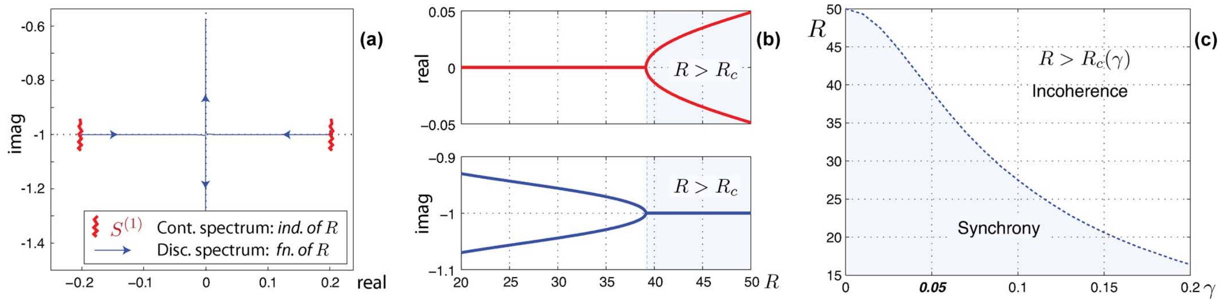

Fig. 2. Spectrum as a function of R. (a) The continuous spectrum for $k = 1$, along with the two eigenvalue paths as R decreases. (b) The real and imaginary parts of the two eigenvalue paths as R decreases. (c) $R_c(\gamma)$ as a function of γ.

each $\omega_0 \in \Omega$, the point $\pm(\sigma^2/2)k^2 - k\omega_0 i \in S_c^{(k)}$ lies in the continuous spectrum. On the complex plane, $S_c^{(k)}$ comprises of two line segments, one in the left half-plane and the other in the right half-plane. The main thing to note is that the continuous spectrum does not change with the value of R and is moreover bounded away from the imaginary axis for $k = \pm 1, \pm 2, \cdots$. So, the focus of the analysis and the numerical study that follows is on the discrete spectrum. As implied by Theorem 4.1, the discrete spectrum is obtained by solving the characteristic equation:

$$\frac{k^2 C_k^{\bullet}}{2R} \int_{\Omega} \frac{g(\omega)}{\left(\lambda - \frac{\sigma^2}{2}k^2 + k\omega i\right)\left(\lambda + \frac{\sigma^2}{2}k^2 + k\omega i\right)} \mathrm{d}\omega - 1 = 0$$

for $k = 1, 2, \cdots$ such that $C_k^{\bullet} \neq 0$. For negative values of k, the eigenvalues are simply the complex conjugate.

Example 1: Consider $c^{\bullet}(\vartheta - \theta) = (1/2)\sin^2((\vartheta - \theta)/2)$. In this case $C_1^{\bullet} = -(1/4)$ and there is only one characteristic equation to consider

$$\frac{1}{8R} \int_{\Omega} \frac{g(\omega)}{\left(\lambda - \frac{\sigma^2}{2} + \omega i\right)\left(\lambda + \frac{\sigma^2}{2} + \omega i\right)} \mathrm{d}\omega + 1 = 0. \quad (27)$$

This equation was solved numerically to obtain a path of eigenvalues as a function of R. In these calculations, we fixed $\sigma^2 = 0.1$ and $g(\omega)$ a uniform distribution on $\Omega = [1 - \gamma, 1 + \gamma]$.

Fig. 2(a) depicts the resulting locus of eigenvalues obtained with $\gamma = 0.05$. For $R \sim \infty$ there are a pair of complex eigenvalues at $\sim \pm(\sigma^2/2) - i$. As the parameter R decreases, these eigenvalues move continuously towards the imaginary axis. The critical value R_c is defined as the value of R at which these two eigenvalue paths collide on the imaginary axis, resulting in an eigenvalue pair of multiplicity 2. The eigenvalues split as R is decreased further, and remain on the imaginary axis for $R < R_c$. The real and the imaginary part of the two eigenvalue paths originating at $\pm(\sigma^2/2) - i$ are depicted in Fig. 2(b). These eigenvalues also have their complex conjugate counterparts (for $k = -1$) that are not depicted for the sake of clarity.

In (a) and (b), the value of γ is fixed at 0.05. The critical value R_c is a function of the parameter γ. Fig. 2(c) depicts a plot of $R_c(\gamma)$ as a function of γ. For the uniform distribution $g(\omega)$, the critical point also has an analytical expression:

$$R_c(\gamma) = \begin{cases} \frac{1}{2\sigma^4} & \text{if } \gamma = 0, \\ \frac{1}{4\sigma^2\gamma} \tan^{-1}\left(\frac{2\gamma}{\sigma^2}\right) & \text{if } \gamma > 0. \end{cases} \quad (28)$$

The formula (28) is consistent with the expression for critical coupling ($\kappa_c(\gamma) = 2\gamma/\tan^{-1}(2\gamma/\sigma^2)$) for the Kuramoto model in [19]. Note here that the optimal control scales as $1/R$ [see (15)]. ■

C. Stability of the Incoherence Solution

To investigate linear asymptotic stability of the incoherence solution $\mathfrak{z}_0 = (\mathfrak{h}_0, \mathfrak{p}_0)$, we consider the linear initial value problem (22), (23).

The analysis is carried out in the Fourier coordinates. Using the diagonal representation (26) the linear evolution equation is given by

$$\frac{d}{dt} Z_k(t, \omega) = \mathcal{L}_R^{(k)} Z_k(t, \omega).$$

We have two cases to consider:

1) $C_k^{\bullet} = 0$: $\mathcal{L}_R^{(k)}$ is a triangular matrix and the solution is given in closed-form

$$Z_k(t, \omega) = e^{\mathcal{L}_R^{(k)} t} Z_k(0, \omega) = \begin{pmatrix} e^{\left(\frac{\sigma^2}{2}k^2 - k\omega i\right)t} & 0 \\ \bullet & e^{\left(-\frac{\sigma^2}{2}k^2 - k\omega i\right)t} \end{pmatrix} \times \begin{pmatrix} H_k(0, \omega) \\ P_k(0, \omega) \end{pmatrix}.$$

The only function in $\mathbf{L}^2$ that satisfies

$$H_k(t, \omega) = e^{\left(\frac{\sigma^2}{2}k^2 - k\omega i\right)t} H_k(0, \omega)$$

is the zero function. We thus have

$$H_k(t, \omega) = 0, \quad P_k(t, \omega) = e^{\left(-\frac{\sigma^2}{2}k^2 - k\omega i\right)t} Q_k(\omega) \quad (29)$$

i.e., $P_k(t, \omega) \to 0$ as $t \to \infty$.

2) $C_k^{\bullet} \neq 0$: We denote $a_k(\omega) := (\sigma^2/2)k^2 + k\omega i$ and $\bar{a}_k(\omega) = (\sigma^2/2)k^2 - k\omega i$. In the following we assume $k \geq 1$ (solution for negative values of k can be obtained as the complex conjugate).

In the Fourier coordinates, the linear evolution equation is

$$\frac{\mathrm{d}H_k}{\mathrm{d}t}(t, \omega) = \bar{a}_k(\omega) H_k(t, \omega) - \pi C_k^{\bullet} \int_{\Omega} P_k(t, \omega) g(\omega) \mathrm{d}\omega,$$

$$\frac{\mathrm{d}P_k}{\mathrm{d}t}(t, \omega) = -\frac{k^2}{2\pi R} H_k(t, \omega) - a_k(\omega) P_k(t, \omega)$$

whose solution is given by

$$H_k(t,\omega) = \pi C_k^\bullet e^{\bar{a}_k(\omega)t} \int_t^\infty e^{-\bar{a}_k(\omega)\tau} \times \int_\Omega P_k(\tau,\omega) g(\omega) \mathrm{d}\omega \mathrm{d}\tau \tag{30}$$

$$P_k(t,\omega) = e^{-a_k(\omega)t} Q_k(\omega) - \frac{k^2 e^{-a_k(\omega)t}}{2\pi R} \int_0^t e^{a_k(\omega)\tau} H_k(\tau,\omega) \mathrm{d}\tau. \tag{31}$$

Substituting (30) in (31), and denoting

$$\mathcal{M}^{(k)} P_k(t,\omega) := -\frac{k^2 C_k^\bullet}{2R} e^{-a_k(\omega)t} \times \int_0^t e^{\sigma^2 \tau} \left(\int_\tau^\infty e^{-\bar{a}_k(\omega)s} \int_\Omega P_k(s,\omega) g(\omega) \mathrm{d}\omega \mathrm{d}s \right) \mathrm{d}\tau$$

we arrive at the fixed-point equation

$$P_k(t,\omega) = e^{-a_k(\omega)t} Q_k(\omega) + \mathcal{M}^{(k)} P_k(t,\omega).$$

The linear initial value problem thus entails analysis of this fixed-point equation for all k with $C_k^\bullet \neq 0$.

We describe the analysis with the aid of Example 1 where $c^\bullet(\vartheta - \theta) = (1/2)\sin^2((\vartheta-\theta)/2))$. In this case, $C_1^\bullet = -(1/4)$ and $C_k^\bullet = 0$ for $k = 2, 3, \ldots$. So, we need consider only the harmonic $k = 1$ whose solution is given by the fixed-point equation

$$P_1(t,\omega) = e^{-\left(\frac{\sigma^2}{2} + \omega i\right)t} Q_1(\omega) + \mathcal{M}^{(1)} P_1(t,\omega) \tag{32}$$

where

$$\mathcal{M}^{(1)} P_1(t,\omega) = \frac{1}{8R} e^{-\left(\frac{\sigma^2}{2} + \omega i\right)t} \int_0^t e^{\sigma^2\tau} \times \left(\int_\tau^\infty e^{-\left(\frac{\sigma^2}{2} - \omega i\right)s} \int_\Omega P_1(s,\omega) g(\omega) \mathrm{d}\omega \mathrm{d}s \right) \mathrm{d}\tau. \tag{33}$$

$\mathcal{M}^{(1)} : \mathbf{L}^2(\mathbb{R}^+, \mathbf{H}_\varpi) \to \mathbf{L}^2(\mathbb{R}^+, \mathbf{H}_\varpi)$ is a linear operator and its Laplace transform is denoted as $\hat{\mathcal{M}}^{(1)}(\lambda,\omega)$, where λ is the Laplace transform variable. The transform is given by

$$\hat{\mathcal{M}}^{(1)}(\lambda,\omega) = \frac{1}{8R} \frac{1}{\left(\lambda + \frac{\sigma^2}{2} + \omega i\right)\left(\lambda - \frac{\sigma^2}{2} + \omega i\right)}.$$

The $\mathbf{L}^2 \to \mathbf{L}^2$ induced operator norm for $\mathcal{M}^{(1)}$ is given by

$$\left\| \hat{\mathcal{M}}^{(1)}(\lambda,\omega) \right\|_\infty = \sup_{\lambda \in \mathbb{I}} \frac{1}{8R} \left| \int_\Omega \frac{1}{\left(\lambda + \frac{\sigma^2}{2} + \omega i\right)\left(\lambda - \frac{\sigma^2}{2} + \omega i\right)} g(\omega) \mathrm{d}\omega \right| \tag{34}$$

where $\mathbb{I}$ denotes the imaginary axis.

For well-posedness, we require $\mathcal{M}^{(1)}$ is a contraction, i.e., $\|\hat{\mathcal{M}}^{(1)}(\lambda,\omega)\|_\infty < 1$. The proof of the following Lemma appears in the Appendix VII-E.

Lemma 4.2: The linear operator $\mathcal{M}^{(1)} : \mathbf{L}^2(\mathbb{R}^+, \mathbf{H}_\varpi) \to \mathbf{L}^2(\mathbb{R}^+, \mathbf{H}_\varpi)$ as defined by (33) is a contraction if and only if the eigenvalues of the characteristic equation (27) are not on the imaginary axis. ∎

In Example 1, we saw that there is a critical value of $R = R_c(\gamma)$ above which the eigenvalues are not on the imaginary axis. For such values of R, we have the following well-posedness conclusion for the linear initial value problem. The proof appears in the Appendix VII-F.

Theorem 4.3: Consider the linear initial value problem (22), (23) with $c^\bullet(\vartheta - \theta) = (1/2)\sin^2((\vartheta - \theta)/2)$. Suppose $R > R_c(\gamma)$, so the roots of the characteristic equation (27) are not on the imaginary axis. Then

1) **Existence and uniqueness**. A unique solution exists in $\mathbf{L}^2(\mathbb{R}^+, \mathbf{H})$ and is given by

$$\tilde{p}(\theta,t;\omega) = P_1(t,\omega)e^{i\theta} + \sum_{k=2}^{+\infty} e^{\left(-\frac{\sigma^2}{2}k^2 - k\omega i\right)t} Q_k(\omega) e^{ik\theta} + c.c.,$$
$$\tilde{h}(\theta,t;\omega) = H_1(t,\omega)e^{i\theta} + c.c.$$

where $P_1(t,\omega)$ is a unique solution of the fixed-point equation (32), and

$$H_1(t,\omega) = -\frac{\pi}{4} e^{\left(\frac{\sigma^2}{2} - \omega i\right)t} \int_t^\infty e^{\left(-\frac{\sigma^2}{2} + \omega i\right)\tau} \times \int_\Omega P_1(\tau,\omega) g(\omega) \mathrm{d}\omega \mathrm{d}\tau.$$

2) **Stability**. As $t \to \infty$, $\tilde{p}(\theta,t;\omega) \to 0$, i.e., the incoherence solution is linearly asymptotically stable.

∎

Remark 2: The linear analysis provides for a game theoretic generalization of the linear stability analysis that first appeared for the Kuramoto model in [19]. Technically, the main difference here is due to the forward–backward nature of the coupled PDE model. This yields spectra for the linear problem that is symmetric about the imaginary axis. As a result, stability cannot be deduced directly in terms of the real part of the spectra alone (as in [19]). The stability analysis instead requires one to show contraction properties of the fixed-point equation (32). A similar construction also appears in [2]. ∎

Remark 3: It is worthwhile to note that the zero noise limit (as $\sigma \downarrow 0$) leads to a singular problem. Although the analysis techniques of this paper are no longer relevant to the analysis of the $\sigma = 0$ problem, it is an important open question on whether the phase transition phenomena occur also for the limiting case? Similar singular problems also arise in many important application areas in mathematical physics. We refer the reader to the review paper [24] for a discussion of the analysis techniques and open problems for the coupled oscillator models. ∎

D. Bifurcation Analysis

The spectral analysis suggests the possibility of a bifurcation leading to periodic solutions below the critical value $R = R_c(\gamma)$ (see Fig. 2). In this section, we describe a bifurcation result for the homogeneous population, under which there is a single frequency:

Assumption $(A\mathbf{1}^*)$*:* The density g is given by $g(\omega) = \delta(\omega - 1)$, i.e., the population is homogeneous with a single frequency $\omega_i = 1$ for $i = 1, \ldots, N$. ∎

For the homogeneous population, we denote the solution of the coupled nonlinear PDEs (14a)–(14c) as $(\mathfrak{p}(\theta,t),\mathfrak{h}(\theta,t))$, where dependence on ω is suppressed because there is only a single frequency $\omega=1$.

The existence result is based on the presence of a certain symmetry group: With a spatially invariant cost function, the nonlinear PDEs (14a)–(14c) are equivariant with respect to the spatial symmetry group $SO(2)$:

$$r_\vartheta\,[p(\theta,t),h(\theta,t)]=[p(\theta+\vartheta,t),h(\theta+\vartheta,t)]\,,\ \text{for}\ \vartheta\in[0,2\pi].$$

In PDEs with $SO(2)$ spatial symmetry, bifurcated periodic solutions are known to arise as traveling waves (see [25]) and we assume an ansatz:

$$\mathfrak{p}(\theta,t)=\mathfrak{p}(\theta-at),\quad \mathfrak{h}(\theta,t)=\mathfrak{h}(\theta-at) \tag{35}$$

where a denotes the wave speed. We note that the argument $\theta-at$ on the right hand-side is evaluated mod 2π here and also in the remainder of this paper.

The study of traveling wave solutions is based on the following nonlinear eigenvalue problem:

$$G(v,\eta;R):=\partial^2_{\theta\theta}v+\frac{2}{\sigma^4R}\left(\eta-\int_0^{2\pi}c^\bullet(\theta,\vartheta)v^2(\vartheta)\mathrm{d}\vartheta\right)v$$
$$=0 \tag{36}$$

$$B(v):=\int v^2(\theta)\mathrm{d}\theta-1=0. \tag{37}$$

The eigenvalue problem is important on account of the following Lemma. The proof appears in the Appendix VII-G.

Lemma 4.4: Consider a homogeneous population with frequency $\omega=1$. Suppose (v,η) is a solution of the nonlinear eigenvalue problem (36), (37). Then a traveling wave solution of the coupled nonlinear PDEs (14a)–(14c) is given by

$$\begin{aligned}\mathfrak{p}(\theta,t)&=v^2(\theta-at),\\ \mathfrak{h}(\theta,t)&=-\frac{\sigma^2R}{2}\ln v^2(\theta-at)\end{aligned} \tag{38}$$

with wave speed $a=\omega=1$ and η is the average cost. Conversely, a traveling wave solution $(\mathfrak{p},\mathfrak{h})$ of (14a)–(14c) with wave-speed $a=\omega=1$ is of the form (38), where (v,η) is a solution of (36), (37). ■

In the following we describe solutions of the nonlinear eigenvalue problem (36), (37). We denote $\mathbf{X}:=C^2_{2\pi}([0,2\pi],\mathbb{R})$, the space of twice continuously differentiable real-valued periodic functions on $[0,2\pi]$, and $\mathbf{Y}:=C^0_{2\pi}([0,2\pi],\mathbb{R})$, $G:\mathbf{X}\times\mathbb{R}^+\to\mathbf{Y}$, and $B:\mathbf{X}\to\mathbb{R}$. For any fixed $R\in\mathbb{R}^+$, we are interested in obtaining solutions $(v,\eta)\in\mathbf{X}\times\mathbb{R}^+$ such that $G(v,\eta,R)=0$ and $B(v)=0$.

For the nonlinear eigenvalue problem, we define the incoherence solution

$$v=v_0:=\frac{1}{\sqrt{2\pi}},\quad \eta=\eta_0:=C^\bullet_0=\frac{1}{2\pi}\int_0^{2\pi}c^\bullet(\theta)\mathrm{d}\theta.$$

About the incoherence solution, the linearization of (36) is given by

$$\mathcal{L}_R\tilde v(\theta):=\partial^2_{\theta\theta}\tilde v-\frac{2}{\sigma^4R}\frac{1}{\pi}\int_0^{2\pi}c^\bullet(\theta,\vartheta)\tilde v(\vartheta)\mathrm{d}\vartheta \tag{39}$$

with $\tilde v\in\mathbf{X}$ and satisfies the integral constraint $\int_0^{2\pi}\tilde v(\theta)\mathrm{d}\theta=0$.

The spectrum of the linear operator $\mathcal{L}_R:\mathbf{X}\to\mathbf{Y}$ is summarized in the following:

Theorem 4.5: Consider the linear eigenvalue problem $\mathcal{L}_Rv=\lambda v$. The spectrum consists of eigenvalues $\lambda=-k^2-(2/\sigma^4R)C^\bullet_k=:\lambda_k$ for $k=0,1,2,\ldots$ The eigenspace for the kth eigenvalue $\lambda=\lambda_k$ is given by $\mathrm{span}\{\cos(k\theta),\sin(k\theta)\}$. ■

As the parameter R varies, the potential bifurcation points are where an eigenvalue crosses zero. The kth such bifurcation point is given by $R=-(2/k^2\sigma^4)C^\bullet_k$. Recall that $C^\bullet_k$ denotes the kth Fourier coefficient of the cost function $c^\bullet$ [see (21)].

Consider now the Example 1 with $c^\bullet(\theta-\vartheta)=(1/2)\sin^2((\vartheta-\theta)/2)$. In this case, $C^\bullet_1=-(1/4)$ and the first bifurcation point $R=1/2\sigma^4=R_c(0)$ is the critical point at which the incoherence solution loses stability [see (28)].

We state the bifurcation result next. The proof appears in the Appendix VII-H.

Theorem 4.6: Consider the nonlinear eigenvalue problem (36), (37) with cost function $c^\bullet(\vartheta-\theta)=(1/2)\sin^2((\vartheta-\theta)/2)$. Let (v_0,η_0) denote the incoherence solution. Then from $R=R_c=(1/2\sigma^4)$ bifurcates a branch of non-constant solutions (v,η) of (36), (37). More precisely, there exists a neighborhood $J\subset\mathbb{R}$ of $x=0$, functions $\hat\eta(x),\hat R(x)\in C^1(J)$, and a family $v(x)$ of non-constant solutions of (36), (37) in $\mathbf{X}$ such that

1) $\eta=\hat\eta(x)$ and $\hat\eta(x)\to\eta_0$, $R=\hat R(x)$ and $\hat R(x)\to R_c$ as $x\to0$;
2) the amplitude of $v(x)-v_0$ tends to zero as $x\to0$.

■

Remark 4: The Lyapunov–Schmidt perturbation method was used to obtain an asymptotic formula for the non-constant bifurcating solution branch. For the cost $c^\bullet(\vartheta-\theta)=(1/2)\sin^2((\vartheta-\theta)/2)$ as in Example 1, the solution is given by an asymptotic formula in the small "amplitude" parameter x:

$$\begin{aligned}v(x)&=v_0+(2\cos(\theta+\theta_0))\,x\\&\quad+\left(-\frac{1}{v_0}+v_0\pi\cos2(\theta+\theta_0)\right)x^2+\mathcal{O}(x^3),\\ \eta&=\hat\eta(x)=\eta_0-\pi x^2+\mathcal{O}(x^3),\\ R&=\hat R(x)=R_c-\frac{7\pi}{2\sigma^4}x^2+\mathcal{O}(x^3)\end{aligned}$$

where $R_c=(1/2\sigma^4)$, $\eta_0=(1/4)$ and θ_0 is an arbitrary phase in $[0,2\pi]$. Fig. 3 depicts the bifurcation diagram for the average cost η as a function of the bifurcation parameter R. For comparison, we also depict the numerical solution of the nonlinear eigenvalue problem that is obtained using the continuation software AUTO [26]. The details of the calculations are omitted on account of space. ■

Using Lemma 4.4, we also have an existence result for traveling wave solutions of the form (38) for the coupled nonlinear PDEs (14a)–(14c).

Corollary 4.1: Consider the coupled nonlinear PDE (14a)–(14c) with cost function $c^\bullet(\vartheta-\theta)=(1/2)\sin^2((\vartheta-\theta)/2)$ and homogeneous frequency $\omega=1$. Let $(\mathfrak{p}_0,\mathfrak{h}_0)$ denote the incoherence solution. Then from $R=R_c=1/2\sigma^4$ bifurcates a branch of traveling wave solutions $(\mathfrak{p},\mathfrak{h})=(v^2(\theta-at),-(\sigma^2R/2\ln v^2(\theta-at))$ with wave-speed $a=\omega=1$, where v are the non-constant solutions as described in Theorem 4.6 and Remark 4. ■

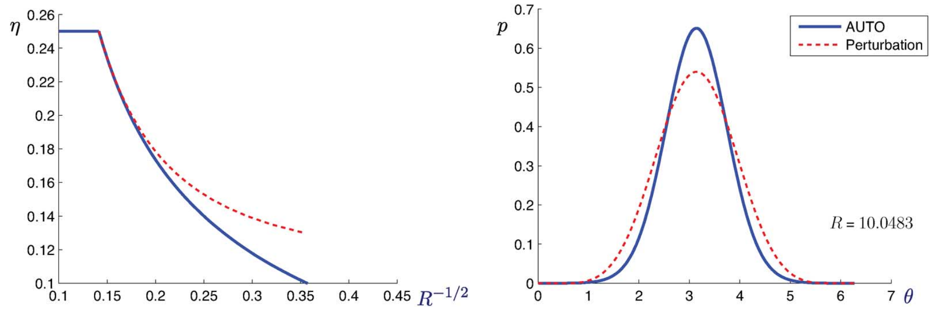

Fig. 3. (a) Bifurcation diagram for the average cost as a function of parameter $1/\sqrt{R}$. (b) The solution $v^2(\theta)$ for $R = 10$ ($R^{-1/2} = 0.31$).

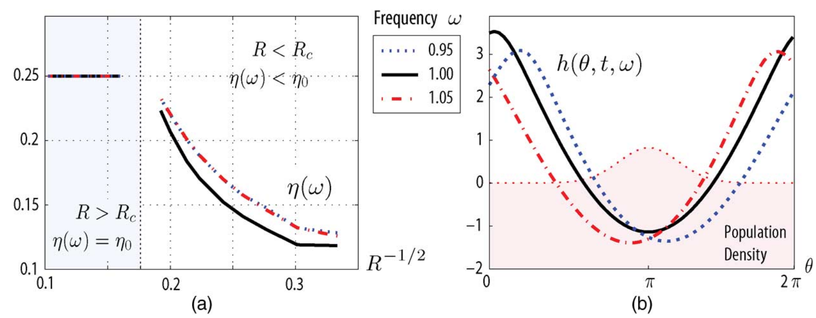

Fig. 4. Numerical results. (a) Bifurcation diagram: the average cost as a function of $1/\sqrt{R}$. (b) Relative value function for $R = 10$, and the population density p for a particular value of t. As t varies, the solution rotates along the circle $[0, 2\pi]$ with a constant wave speed 1.

V. Numerical Results

We present here numerically obtained solutions of the coupled nonlinear PDEs (14a)–(14c) with heterogeneous frequencies, and cost function $c^\bullet(\vartheta - \theta) = (1/2)\sin^2((\vartheta - \theta)/2)$ as in Example 1. We fix $\sigma^2/2 = 0.05$ and $\gamma = 0.05$ which gives $R_c(\gamma) = 39.1$. The computations that follow are based on a waveform relaxation algorithm, whose details can be found in the conference version of this paper [1].

In numerical experiments, the uniform distribution $g(\omega) = (2\gamma)^{-1}$ on the interval $\Omega = [1 - \gamma, 1 + \gamma]$ is approximated by a uniform distribution on three discrete frequencies $\{1 - \gamma, 1, 1 + \gamma\}$. The value of $\gamma = 0.05$ is sufficiently small so that the numerical results are very similar to those obtained using a finer discretization of Ω. The PDEs are discretized along the θ coordinate using the method of Fourier collocation, with 64 collocation points in the interval $[0, 2\pi]$.

A. Average Cost Bifurcation Diagram

Fig. 4(a) depicts the bifurcation diagram for the average cost $\eta(\omega)$ as a function of the bifurcation parameter R.

For $R > R_c = 39.1$, the average cost was found to be $\eta(\omega) = \eta_0 = (1/4)$, which is consistent with the incoherence solution of Section II-D. For $R < R_c$ the average cost is reduced, and for such R the value of $\eta(\omega) < \eta_0$ depends upon the frequency ω. Its minimal value is attained uniquely when $\omega = 1$, which is the mean frequency under g.

B. Value Functions, Control, and Density Evolution

The relative value function $h(\theta, t; \omega)$ and probability density $p(\theta, t; \omega)$ were computed for a range of values of R.

The incoherence solution $h_0 \equiv 0$ was obtained for $R \geq 60$; the algorithm was very slow to converge as R was reduced to values near R_c.

Fig. 4(b) depicts the relative value function as a function of θ obtained for $R = 10 < R_c$, and for a particular value of t. Experiments revealed that the relative value function and the solution to the FPK equation arise as a traveling wave solution. In particular, the solution has the form

$$\mathfrak{p}(\theta, t; \omega) = \mathfrak{p}(\theta - at, 0; 1), \quad \mathfrak{h}(\theta, t; \omega) = \mathfrak{h}(\theta - at, 0; \omega).$$

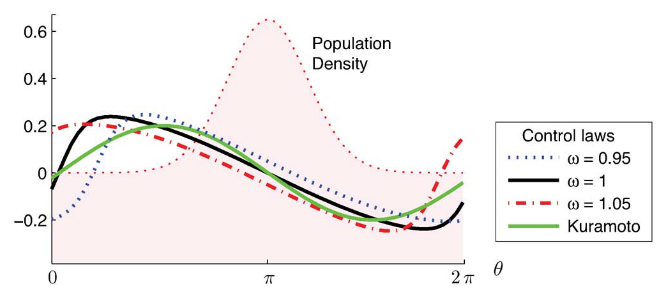

Fig. 5. Comparison of the control obtained from solving (14a)–(14c) and the Kuramoto model.

Moreover, the wave speed was equal to $a = 1$, independent of ω, which coincides with the mean frequency with respect to the density g.

Recall the control law (15) is $u_i^o(t) = -(1/R)\partial_\theta h(\theta, t; \omega)|_{\omega=\omega_i}$, which depends upon the frequency $\omega = \omega_i$. The control laws obtained for a fixed t and several values of ω are depicted in Fig. 5 in relation to the population density. Note that the control is zero when $\omega = 1$, and θ lies at its mean value (equal to π in this figure, for the particular value of t chosen).

The control law that gives rise to the Kuramoto oscillator is defined by $u_i^{(\mathrm{Kur})}(\theta_i, t) = (\kappa/N)\sum_{j=1}^N \sin(\theta_j(t) - \theta_i)$. Given the previous numerical results using $R = 10$, it is reasonable to conjecture that as N tends to infinity this can be approximated by $u_i^{(\mathrm{Kur})}(\theta_i, t) = \kappa_0 \sin(\vartheta_0 + t - \theta_i)$, for a phase variable ϑ_0 and a gain κ_0 that is proportional to κ. This is because in the synchrony state, the individual oscillators rotate with a common frequency 1. That is, $\theta_j(t) \approx t + \vartheta_{0,j}$ for some $\vartheta_{0,j} \in [0, 2\pi]$. Fig. 5 shows that the optimal control law is in fact "close" to $u_i^{(\mathrm{Kur})}$ when κ_0, t, and ϑ_0 are chosen appropriately. A detailed comparison between the game theoretic and Kuramoto control laws appears in [27].

VI. CONCLUSION

This paper aggregates concepts and techniques from nonlinear dynamical systems, stochastic control, game theory, and statistical mechanics to provide new tools for understanding complex interconnected systems, and new bridges with prior research. The key messages are as follows.

1) Distributed control laws are tractable for a class of large population dynamic games with separable cost structures. This conclusion is based on an approximation of the complex stochastic system using a deterministic PDE model, similar to the mean-field approximation that is central to the study of interacting particle systems.
2) The rich theory surrounding the classical Kuramoto model can be extended to the dynamic game setting introduced here to explain phase transitions in these systems. In particular, methods from bifurcation theory can be adopted to analyze multiple equilibria and their stability properties.

The future work will focus on applications of proposed models to problems in neuroscience: in particular, on development of adaptation algorithms for "learning" approximately optimal control laws [27]. Relevance of such architectures to established learning paradigms (e.g., long-term potentiation (LTP) that underlies synaptic plasticity [5]) in neuroscience will be investigated.

Another possible direction is further analysis of the solutions of the coupled PDE model: in particular, investigation of stability and possible bifurcation of the time-periodic synchrony solution. Non-periodic solutions, if they exist, of the coupled PDE model will invite further research on ϵ-Nash optimality of a general class of solutions.

APPENDIX

A. Proof of Proposition 2.1

The proof of 1) is standard, even in this nonstandard setting in which $\bar{c}$ varies arbitrarily with time. For any input we define the stochastic process

$$M(T) = h_i(\theta_i(T),T) + \int_0^T \left(\bar{c}(\theta_i(t),t) + \frac{1}{2}Ru_i^2(t) - \eta_i^*\right) dt, \quad T \geq 0.$$

This is a martingale under the Markov policy (9), and a submartingale under any other input. In fact, under (9) we have

$$M(T) = h_i(\theta_i(0),0) + \sigma \int_0^T \partial_\theta h_i(\theta_i(t),t)\, d\xi_i(t).$$

While, for any other input this is an inequality. The Dambis–Dubins–Schwarz Theorem (given as Theorem 4.6 of [28]) implies that there is a Brownian motion $\hat{\xi}$ such that the following holds for all $T \geq 0$

$$\int_0^T \partial_\theta h_i(\theta_i(t),t)\, d\xi_i(t) = \hat{\xi}_i(\delta(T))$$

where $\delta(T) = \int_0^T (\partial_\theta h_i(\theta_i(t),t))^2 dt$. The uniform bound imposed on $\partial_\theta h_i$ implies that $M(T)/T \to 0$, as $T \to \infty$, with probability one under (9). This establishes average-cost optimality under the boundedness assumptions on h_i and its derivatives.

B. Proof of Proposition 3.4

We denote by $\theta_j^o(t)$ the solution to the SDE (3), obtained using the oblivious control (15). Denote $\bar{c}^N(\theta_i(s),s) = N^{-1}\sum_{j\neq i} c^\bullet(\theta_i(s),\theta_j^o(s))$. Prop. 3.3 implies that the Law of Large Numbers holds: For each $N \geq 1$ there exists a limit η_N, depending only on $\{\omega_i : 1 \leq i \leq N\}$ and φ^N such that

$$\eta_N = \lim_{T\to\infty} \frac{1}{T}\int_0^T \bar{c}^N(\theta_i(s),s)\, ds. \tag{40}$$

A limit also holds for the averaged cost function:

$$\eta_N^o := \lim_{T\to\infty} \frac{1}{T}\int_0^T \bar{c}(\theta_i(s),s)\, ds \tag{41}$$

which depends only on φ^N and ω_i. Our goal is to demonstrate that $\eta_N - \eta_N^o$ vanishes as $N \to \infty$.

Letting π_0^N denote the invariant distribution for the skeleton chain with $r = 0$, we have the representations

$$\eta_N = \frac{1}{N}\sum_{j\neq i}\frac{1}{\tau}\int_0^\tau \mathsf{E}_{\pi_0^N}\left[c^\bullet\left(\theta_i(s),\theta_j^o(s)\right)\right] ds \tag{42}$$

$$\eta_N^o = \frac{1}{\tau}\int_0^\tau \mathsf{E}_{\pi_0^N}\left[\bar{c}(\theta_i(s),s)\right] ds. \tag{43}$$

Each of the expectations in (42) is *conditional* on the frequencies $\{\omega_i : 1 \leq i \leq N\}$.

To identify the limit of $\{\eta_N - \eta_N^o\}$ as $N \to \infty$ we first consider the average in (42) in the simpler situation in which $\theta_i(t)$ is fixed: We denote

$$\mathcal{C}(\vartheta,\omega_j) = \frac{1}{\tau}\int_0^\tau \mathsf{E}_{\pi_0^N}\left[c^\bullet\left(\vartheta,\theta_j^o(s)\right)\right] ds$$

where the expectation is independent of the feedback law φ^N (recall that $\{\theta_j(t) : j \neq i\}$ are mutually independent since we have assumed (15)) and independent initial conditions [see Assumption (A1)]. We then have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j\neq i}\mathcal{C}(\vartheta,\omega_j) = \int \mathcal{C}(\vartheta,\omega)g(\omega)d\omega = \frac{1}{\tau}\int_0^\tau \bar{c}(\vartheta,t)dt$$

where the first identity follows from the LLN for the i.i.d. sequence of frequencies, and the second equality is the definition of $\bar{c}$. Moreover, under the assumption that $\bar{c}$ is continuous, the family of functions on the left-hand side is *equicontinuous* on $[0,2\pi]$. It follows that the convergence is uniform:

$$\lim_{N\to\infty}\sup_\vartheta\left|\frac{1}{N}\sum_{j\neq i}\mathcal{C}(\vartheta,\omega_j) - \frac{1}{\tau}\int_0^\tau \bar{c}(\vartheta,t)dt\right| = 0. \tag{44}$$

Returning to (42) and (43), we obtain the desired conclusion:

$$\lim_{N\to\infty}|\eta_N - \eta_N^o| = \lim_{N\to\infty}\left|\mathsf{E}_{\pi_0^N}\left[\frac{1}{N}\sum_{j\neq i}\frac{1}{\tau}\int_0^\tau c^\bullet\left(\theta_i(s),\theta_j^o(s)\right) - \bar{c}(\theta_i(s),s)\, ds\right]\right|$$

$$= \lim_{N\to\infty}\left|\mathsf{E}_{\pi_0^N}\left[\frac{1}{N}\sum_{j\neq i}\mathcal{C}\left(\theta_i(s),\omega_j\right) - \frac{1}{\tau}\int_0^\tau \bar{c}\left(\theta_i(s),s\right)ds\right]\right|$$
$$\leq \lim_{N\to\infty}\sup_{\vartheta}\left|\frac{1}{N}\sum_{j\neq i}\mathcal{C}(\vartheta,\omega_j) - \frac{1}{\tau}\int_0^\tau \bar{c}(\vartheta,s)ds\right| = 0.$$

This establishes convergence with probability one.

To establish the rate of convergence in mean square (19) we consider (42) and (43) from a different perspective. For fixed N, the $N-1$ random variables $\{\tilde{\mathcal{C}}_j := \mathsf{E}_{\pi_0^N}[c^\bullet(\theta_i(s),\theta_j^o(s))] - \eta_N^o\}$ are i.i.d. with zero mean. They are also uniformly bounded by $2\|c^\bullet\|_\infty$ (twice the maximum of $c^\bullet$ over θ and t). Consequently,

$$\mathsf{E}\left[\left(\frac{N}{N-1}\eta_N - \eta_N^o\right)^2\right] = \mathsf{E}\left[\left(\frac{1}{N-1}\sum_{j\neq i}\tilde{\mathcal{C}}_j\right)^2\right] \leq \frac{4\|c^\bullet\|_\infty^2}{N-1}.$$

This implies the absolute bound, $\limsup_{N\to\infty} N\mathsf{E}[\epsilon_N^2] \leq 4\|c^\bullet\|_\infty^2$.

C. Proof of Theorem 3.5

We first prove the result for the case where the ith oscillator control is of the state-feedback form $u_i(t) = \varphi^N(\theta_i(t),\theta_{-i}(t),[t]_\tau)$ where the function $\varphi^N : \mathsf{X} \to \mathbb{R}$ is continuous.

Using the definition (17) of the average cost

$$\begin{aligned}
&\eta_i^{(\mathrm{POP})}\left(u_i;u_{-i}^o\right)\\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\frac{1}{N}\sum_{j\neq i}c^\bullet\left(\theta_i(s),\theta_j^o(s)\right) + \frac{1}{2}Ru_i^2\right]\mathrm{d}s\\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\bar{c}\left(\theta_i(s),s\right) + \frac{1}{2}Ru_i^2(s)\right]\mathrm{d}s\\
&\quad + \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\frac{1}{N}\sum_{j\neq i}c^\bullet\left(\theta_i(s),\theta_j^o(s)\right) - \bar{c}\left(\theta_i(s),s\right)\right]\mathrm{d}s\\
&= \eta(u_i;\bar{c})\\
&\quad + \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\frac{1}{N}\sum_{j\neq i}c^\bullet\left(\theta_i(s),\theta_j^o(s)\right) - \bar{c}\left(\theta_i(s),s\right)\right]\mathrm{d}s\\
&:= \eta(u_i;\bar{c}) + I_1.
\end{aligned}\tag{45}$$

Similarly,

$$\begin{aligned}
&\eta_i^{(\mathrm{POP})}\left(u_i^o;u_{-i}^o\right)\\
&\quad = \eta\left(u_i^o;\bar{c}\right)\\
&\qquad + \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\frac{1}{N}\sum_{j\neq i}c^\bullet\left(\theta_i^o(s),\theta_j^o(s)\right)\right.\\
&\qquad\qquad \left. - \bar{c}\left(\theta_i^o(s),s\right)\right]\mathrm{d}s\\
&\quad := \eta\left(u_i^o;\bar{c}\right) + I_2.
\end{aligned}\tag{46}$$

Combining (45) and (46)

$$\begin{aligned}
\eta_i^{(\mathrm{POP})}\left(u_i;u_{-i}^o\right) &= \eta(u_i;\bar{c}) + I_1 \geq \eta_i\left(u_i^o;\bar{c}\right) + I_1\\
&= \eta_i^{(\mathrm{POP})}\left(u_i^o;u_{-i}^o\right) - I_2 + I_1\\
&\geq \eta_i^{(\mathrm{POP})}\left(u_i^o;u_{-i}^o\right) - (|I_1| + |I_2|)
\end{aligned}\tag{47}$$

where the first inequality is due to (20). The result follows by using the estimate shown in Proposition 3.4.

We now consider the case where the ith oscillator uses any admissible policy, that is not necessarily periodic state-feedback. The result in this case follows from using (47) together with 3) in Prop. 3.2. Since, the optimal policy for ith oscillator, $u_i(t) = \varphi^*(\theta_i(t),\theta_{-i}(t),[t]_\tau) := u_i^*(t)$, is itself a continuous periodic state-feedback, we have using (47):

$$\eta_i^{(\mathrm{POP})}\left(u_i^*;u_{-i}^o\right) \geq \eta_i^{(\mathrm{POP})}\left(u_i^o;u_{-i}^o\right) - (|I_1| + |I_2|).$$

With any admissible control, $\eta_i^{(\mathrm{POP})}(u_i;u_{-i}^o) \geq \eta_i^{(\mathrm{POP})}(u_i^*;u_{-i}^o)$ and the estimate follows for the general case.

D. Proof of Theorem 4.1

A $\lambda \in \mathbb{C}$ is in the spectrum of $\mathcal{L}_R^{(k)}$ if the inverse $[I\lambda - \mathcal{L}_R^{(k)}]^{-1}$ does not exist as a bounded linear operator on $\mathbf{H}_\varpi \times \mathbf{H}_\varpi$.

1) **Continuous spectrum**: We provide a proof of continuous spectrum only for $C_k^\bullet = 0$. The proof for $C_k^\bullet \neq 0$ is conceptually similar but some of the calculations are a bit more involved. We consider the equation

$$\left(\lambda I - \mathcal{L}_R^{(k)}\right)\begin{pmatrix}H(\omega)\\P(\omega)\end{pmatrix} = \begin{pmatrix}\upsilon(\omega)\\\zeta(\omega)\end{pmatrix}$$

where $\upsilon(\omega), \zeta(\omega) \in \mathbf{H}_\varpi$. Explicitly, this gives

$$\begin{aligned}
\left(\lambda - \frac{\sigma^2}{2}k^2 + k\omega i\right)H(\omega) &= \upsilon(\omega),\\
\left(\lambda + \frac{\sigma^2}{2}k^2 + k\omega i\right)P(\omega) &= -\frac{1}{2\pi R}H(\omega) + \zeta(\omega).
\end{aligned}$$

Formally the inverse, if it exists, is given by

$$H(\omega) = \frac{1}{\lambda - \frac{\sigma^2}{2}k^2 + k\omega i}\upsilon(\omega),\tag{48}$$
$$P(\omega) = \frac{1}{\lambda + \frac{\sigma^2}{2}k^2 + k\omega i}\left[-\frac{1}{2\pi R}H(\omega) + \zeta(\omega)\right].\tag{49}$$

The proof that $\mathcal{L}_R^{(k)}$ is 1-1 for all $\lambda \in \mathbb{C}$ is now straightforward. If $\upsilon(\omega) = 0$ then $H(\omega) = 0$ in $\mathbf{H}_\varpi$ using (48) and if additionally $\zeta(\omega) = 0$ then $P(\omega) = 0$ in $\mathbf{H}_\varpi$ using (49).

Using the formula for the inverse, the inverse operator is bounded if and only if $\lambda \notin S_c^{(k)}$. If $\lambda = \lambda_0 = (\sigma^2/2)k^2 - k\omega_0 i \in S_c^{(k)}$ for some $\omega_0 \in \Omega$, then $\lambda_0 - (\sigma^2/2)k^2 + k\omega i = 0$ for $\omega = \omega_0$ and the inverse $(\lambda_0 - (\sigma^2/2)k^2 + k\omega i)^{-1}$ in (48) is not bounded. The converse also follows similarly.

Finally, the range of $\lambda_0 I - \mathcal{L}_R^{(k)}$ is dense in $\mathbf{H}_\varpi$: e.g., the space of C^1 functions with $\upsilon(\omega_0) = \upsilon'(\omega_0) = 0$, is a dense subset.

2) **Discrete spectrum**: For each $k \geq 1$, let $(H_k, P_k)^{\mathrm{T}}$ denote the eigenvector corresponding to an eigenvalue λ. We assume $\lambda \notin S_c^{(k)}$, the set contained in continuous spectrum. We have

$$\lambda H_k(\omega) = \left(\frac{\sigma^2}{2}k^2 - k\omega i\right) H_k(\omega) - \pi C_k^\bullet \int_\Omega P_k(\omega) g(\omega) \mathrm{d}\omega, \tag{50}$$

$$\lambda P_k(\omega) = -\frac{k^2}{2\pi R} H_k(\omega) - \left(\frac{\sigma^2}{2}k^2 + k\omega i\right) P_k(\omega). \tag{51}$$

We formally obtain from (51):

$$P_k(\omega) = -\frac{k^2}{2\pi R}\frac{H_k(\omega)}{\lambda + \frac{\sigma^2}{2}k^2 + k\omega i}$$

and on substituting this into (50):

$$H_k(\omega) = -\frac{k^2 C_k^\bullet}{2R\left(\lambda - \frac{\sigma^2}{2}k^2 + k\omega i\right)} \int_\Omega \frac{H_k(\omega) g(\omega)}{\lambda + \frac{\sigma^2}{2}k^2 + k\omega i} \mathrm{d}\omega. \tag{52}$$

The solution $H_k, P_k \in \mathbf{H}_\varpi$ because $\lambda \notin S_c^{(k)}$, i.e., $\lambda \pm \sigma^2/2k^2 + k\omega i \neq 0$ for all values of $\omega \in \Omega$. Denote $b := \int_\Omega (H_k(\omega)/(\lambda + (\sigma^2/2)k^2 + k\omega i) g(\omega) \mathrm{d}\omega$ which is a constant independent of ω. This gives $H_k(\omega) = -bC_k^\bullet k^2 (2R(\lambda - (1/2)\sigma^2 k^2 + k\omega i))^{-1}$. Substituting this into (52) yields the characteristic equation for λ:

$$\frac{k^2 C_k^\bullet}{2R} \int_\Omega \frac{g(\omega)}{\left(\lambda - \frac{\sigma^2}{2}k^2 + k\omega i\right)\left(\lambda + \frac{\sigma^2}{2}k^2 + k\omega i\right)} \mathrm{d}\omega - 1 = 0. \tag{53}$$

For $k = -1, -2, \ldots$, the eigenvalue is complex conjugate $\bar{\lambda}$.

E. Proof of Lemma 4.2

Comparison of the characteristic equation (27) and the induced norm (34) reveal that there is an eigenvalue $\lambda = i\lambda_I$ on the imaginary axis if and only if

$$\int_\Omega \hat{M}_1(i\lambda_I, \omega) g(\omega) \mathrm{d}\omega = -1.$$

It immediately follows that if $\|\hat{M}_1(\lambda, \omega)\|_\infty < 1$ then there is no eigenvalue on the imaginary axis. The converse is true because $\int_\Omega \hat{M}_1(i\lambda_I, \omega) g(\omega) \mathrm{d}\omega$ is a continuous non-positive function of λ_I, and furthermore its value is zero for $\lambda_I \to \infty$.

F. Proof of Theorem 4.3

We consider the Fourier representation (24) of the solution. For $k \neq \pm 1$, the Fourier coordinates $H_k = 0$, $P_k = e^{(-(\sigma^2/2)k^2 - k\omega i)t}$ as shown in (29). For $k = 1$, the Fourier coordinate $P_1(t, \omega)$ is a solution of the fixed-point equation (32):

$$P_1(t, \omega) = e^{-\left(\frac{\sigma^2}{2} + i\omega\right)t} Q_1(\omega) + \mathcal{M}^{(1)} P_1(t, \omega).$$

If the roots of the characteristic equation are not on the imaginary axis, then $\mathcal{M}^{(1)}$ is a contraction (using Lemma 4.2) and there exists a unique solution

$$P_1(t, \omega) = \left(I - \mathcal{M}^{(1)}\right)^{-1} e^{-\left(\frac{\sigma^2}{2} + i\omega\right)t} Q_1(\omega)$$

and using (30)

$$H_1(t, \omega) = -\frac{\pi}{4} e^{\left(\frac{\sigma^2}{2} - \omega i\right)t} \int_t^\infty e^{\left(-\frac{\sigma^2}{2} + \omega i\right)\tau} \times \int_\Omega P_1(\tau, \omega) g(\omega) \mathrm{d}\omega \mathrm{d}\tau.$$

G. Proof of Lemma 4.4

For the traveling wave solution [see ansatz (35)], we assume wave speed $a = \omega = 1$. This implies that the left-hand side of the FPK and the HJB PDEs

$$\partial_t p + \omega \partial_\theta p = 0, \quad \partial_t p + \omega \partial_\theta p = 0. \tag{54}$$

We denote $u^* = -(1/R)\partial_\theta h$ and using (54), the FPK equation (14b) is given by

$$\partial_\theta [p u^*] = \frac{\sigma^2}{2} \partial^2_{\theta\theta} p,$$
$$\therefore, u^* = \frac{\sigma^2}{2} \partial_\theta \ln(p) + \frac{K}{p} \tag{55}$$

where K is the constant of integration. Now, $u^* = -(1/R)\partial_\theta h$ where h is periodic. So, $\int_0^{2\pi} u^* \mathrm{d}\theta = 0$. Integrating both sides of (55) over $[0, 2\pi]$, we have

$$0 = \int_0^{2\pi} u^* \mathrm{d}\theta = K \int_0^{2\pi} \frac{1}{p} \mathrm{d}\theta$$

, i.e., $K = 0$ and

$$u^* = \frac{\sigma^2}{2} \partial_\theta \ln(p). \tag{56}$$

Using (54), the HJB equation (14a) is given by

$$-\frac{1}{2R}(\partial_\theta h)^2 + \frac{\sigma^2}{2} \partial^2_{\theta\theta} h = \eta^*(\omega) - \bar{c} \tag{57}$$

where $\bar{c}(\cdot) = \int_0^{2\pi} c^\bullet(\cdot, \vartheta) p(\vartheta) \mathrm{d}\vartheta$. We introduce the Hopf–Cole transformation coordinate v as

$$v = \exp\left(-\frac{1}{\sigma^2 R} h\right) \tag{58}$$

to simplify the HJB equation (57) to $-\partial^2_{\theta\theta} v = (2/\sigma^4 R)(\eta^* - \bar{c})v$.

Finally, using (56) and (58), we obtain $(\sigma^2/2)\partial_\theta \ln(p) = u^* = -(1/R)\partial_\theta h = \sigma^2 \partial_\theta \ln(v)$. This gives $p = v^2$, where we have dropped the constant of integration because h is defined only up to a constant. We thus obtain the eigenvalue problem expressed only in terms of v with the constraint that $\int v^2(\theta) \mathrm{d}\theta = 1$ because $p = v^2$ is a density.

H. Proof of Theorem 4.6

The proof follows closely the Hopf bifurcation theorem using the method of Lyapunov–Schmidt; cf. [25]. We outline below the main steps where some of the calculations from the proof of the Hopf bifurcation will be assumed here.

1) **Symmetry**: The nonlinear operator G is equivariant with respect to the spatial symmetry group $O(2)$:

$$SO(2) : \tau_\vartheta\,[v(\theta)] = v(\theta+\vartheta), \text{ for } \vartheta \in [0, 2\pi]$$
$$Z^2 : \sigma\,[v(\theta)] = v(-\theta).$$

Note that the rotation symmetry $SO(2)$ arises because the convolution kernel $c^\bullet$ is spatially invariant, and reflection symmetry Z^2 is because $c^\bullet$ is assumed to be an even function (see Assumption (A4) in Section IV).

The $O(2)$-equivariance allows us to look for solutions with respect to even (or odd) functions $v(\theta)$. In particular, denote $\mathbf{X}^e := \{v \in X : v(\theta) = v(-\theta)\}$ and similarly for $\mathbf{Y}^e$. Then by equivariance, $G : \mathbf{X}^e \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbf{Y}^e$.

2) **Linear analysis**: About the incoherence solution v_0, the linearization is given by (39). We consider the restriction of the linear operator $\mathcal{L}_R$ to the space of even functions $\mathbf{X}^e$. The eigenvalues are still given by $\lambda = \lambda_k = -k^2 - (2/\sigma^4 R)C_k^\bullet$ for $k = 0, 1, 2, \ldots$. The eigenspace for the kth eigenvalue $\lambda = \lambda_k$ is now one-dimensional, and is given by $\mathrm{span}\{\cos(k\theta)\}$. We consider the eigenvalue $\lambda = \lambda_1 = -1 - (2/\sigma^4 R)C_1^\bullet$ with eigenspace $\mathrm{span}\{\cos(\theta)\}$. The eigenvalue $\lambda = 0$ for the critical parameter value $R = 1/2\sigma^4 =: R_c$.

It is easily verified that the operator $\mathcal{L}_R$ is self-adjoint with respect to the standard inner product

$$\langle v, w\rangle = \frac{1}{\pi}\int v(\theta)w(\theta)\mathrm{d}\theta$$

where $v, w \in \mathbf{L}^2([0, 2\pi])$ are assumed to be periodic functions in domain of the operator. The operator is self-adjoint because the convolution kernel $c^\bullet$ is spatially invariant and even.

3) **Integral constraint**: Suppose $v \in \mathbf{X}^e$ is any solution of the nonlinear eigenvalue problem (36). Integrate over $[0, 2\pi]$ to obtain

$$0 = \eta - C_0^\bullet \int v^2(\theta)\mathrm{d}\theta - \frac{1}{\int v(\theta)\mathrm{d}\theta} \times \sum_k C_k^\bullet \int v(\theta)\cos(k\theta)\mathrm{d}\theta \int v^2(\theta)\cos(k\theta)\mathrm{d}\theta.$$

As a result, the constraint (37) $\int v^2(\theta)\mathrm{d}\theta - 1 = 0$ can be replaced by an equivalent constraint:

$$B'(v,\eta) := \eta - C_0^\bullet - \frac{1}{\int v(\theta)\mathrm{d}\theta}\sum_k C_k^\bullet \int v(\theta)\cos(k\theta)\mathrm{d}\theta \times \int v^2(\theta)\cos(k\theta)\mathrm{d}\theta = 0. \tag{59}$$

4) **Lyapunov-Schmidt reduction**: We consider $R = R_c$ and denote $\mathcal{L}_c = \mathcal{L}_R|_{R=R_c}$. We denote $\mathbf{N} = \ker(\mathcal{L}_c) = \mathrm{span}\{\phi\}$ where $\phi := \cos(\theta)$. Because $\mathcal{L}_c$ is self-adjoint, the range of $\mathcal{L}_c$ is given by $\mathbf{R}(\mathcal{L}_c) = \{y \in \mathbf{Y}^e :< y, \phi >= 0\}$. We consider a direct-sum decomposition $\mathbf{X}^e = \mathbf{N} \oplus \mathbf{M}$ where $\mathbf{M} := \{v \in \mathbf{X}^e :< v, \phi >= 0\}$, and similarly $\mathbf{Y}^e = \mathbf{N} \oplus \mathbf{R}(\mathcal{L}_c)$. By expressing $v = v_0 + x\cos(\theta) + w$ where $w \in \mathbf{M}$ and defining a projection operator $P : \mathbf{Y}^e \to \mathbf{N}$ by $Py :=< y, \phi > \phi$, we express the operator equation $G(v, \eta, R) = 0$ as

$$PG(v_0 + x\cos(\theta) + w, \eta, R) = 0 \tag{60}$$
$$(I - P)G\,(v_0 + x\cos(\theta) + w, \eta, R) = 0. \tag{61}$$

Now, $\mathcal{L}_c : \mathbf{M} \to \mathbf{R}(\mathcal{L}_c)$ is invertible and using implicit function theorem, we have a unique solution $w = \hat{w}(x, \eta, R)$ of (61) in some local neighborhood of $(0, \eta_0, R_c) \subset \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$. Substituting this solution in (60), we obtain a scalar equation

$$s_1(x, \eta, R) := PG\,(v_0 + x\cos(\theta) + \hat{w}(x, \eta, R), \eta, R) = 0. \tag{62}$$

and the constraint gives another scalar equation

$$s_2(x, \eta, R) := B'\,(v_0 + x\cos(\theta) + \hat{w}(x, \eta, R), \eta) = 0. \tag{63}$$

Note by construction there exists a trivial solution $s_1(0, \eta_0, R) = 0$ and $s_2(0, \eta_0, R) = 0$ for all $R \in \mathbb{R}^+$ and in particular for $R = R_c$. Our objective then is to obtain nontrivial solutions of (62) and (63) in terms of the three unknowns (x, η, R).

5) **Scalar equation** (62): The implicit solution of $s_1(x, \eta, R) = 0$ follows using standard Hopf bifurcation calculations; cf. [25]. In particular, $O(2)$ symmetry implies a representation $s_1(x, \eta, R) = \tilde{s}_1(x^2, \eta, R)x$ and a direct calculation shows that

$$\begin{aligned}\frac{\partial \tilde{s}_1}{\partial R}&(0, \eta_0, R_c)\\ &= \frac{\partial^2 s_1}{\partial x \partial R}(0, \eta_0, R_c)\\ &= \left\langle \phi, \frac{\partial^2}{\partial x \partial R} G(v_0 + x\phi + w, \eta, R)\right\rangle\Bigg|_{x=0, \eta=\eta_0, R=R_c}\\ &= \left\langle \phi, \frac{\mathrm{d}}{\mathrm{d}R}\mathcal{L}_R\phi\right\rangle\Bigg|_{R=R_c}\\ &= \frac{d\lambda_1}{dR}(R_c)\langle\phi, \phi\rangle = \frac{2}{\sigma^4 R_c^2}C_1^\bullet \neq 0.\end{aligned}$$

By using implicit function theorem, we obtain a unique solution $R = \hat{R}(x^2, \eta)$ of the scalar equation (62) in some local neighborhood of $(0, \eta_0) \subset \mathbb{R} \times \mathbb{R}^+$.

6) **Scalar equation** (63): By substituting $v = v_0 + x\cos(\theta) + \hat{w}(x, \eta, \hat{R}(x^2, \eta))$ to directly evaluate $B'(v, \eta)$ using (59), we obtain a representation $s_2(x, \eta, R) = \tilde{s}_2(x^2, \eta)$ and direct calculation shows

$$\frac{\partial \tilde{s}_2}{\partial \eta}(0, \eta_0) = 1.$$

By using implicit function theorem again, we obtain a unique solution $\eta = \hat{\eta}(x^2)$ of the scalar equation (63) in some local neighborhood of $0 \subset \mathbb{R}$.

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Huibing Yin (M'10) received the B.E. degree in automatic control from the University of Science and Technology of China, Hefei, in 2003 and the M.E. degree in signal and information processing from the Shanghai Institute of Technical Physics, Chinese Academy of Sciences, Shanghai, in 2006. He is currently pursuing the Ph.D. degree in the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign.

His research interests are in the analysis of nonlinear dynamics and optimization of interconnected and multiagent systems.

Prashant G. Mehta (M'06) received the Ph.D. degree in applied mathematics from Cornell University, Ithaca, NY, in 2004.

He is an Associate Professor in the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign. Prior to joining Illinois, he was a Research Engineer at the United Technologies Research Center (UTRC). His research interests are at the intersection of dynamical systems and control theory, including mean-field games, model reduction, and nonlinear control.

Dr. Mehta has received several awards including an Outstanding Achievement Award for his research contributions at UTRC, several Best Paper awards with his students at Illinois, and numerous teaching and advising honors at Illinois.

Sean P. Meyn (M'87–SM'95–F'02) received the B.A. degree in mathematics (*summa cum laude*) from the University of California, Los Angeles (UCLA) in 1982 and the Ph.D. degree in electrical engineering from McGill University, Montreal, QC, Canada, in 1987 (with Prof. P. Caines).

He held a two year postdoctoral fellowship at the Australian National University in Canberra, and was a Professor at the Department of Electrical and Computer Engineering, and the Coordinated Science Laboratory, at the University of Illinois from 1989 to 2011. Since January, 2012 he has been a Professor and has held the Robert C. Pittman Eminent Scholar Chair in Electrical and Computer Engineering at the University of Florida, Gainesville. He has held visiting positions at universities all over the world, including the Indian Institute of Science, Bangalore, where he was a Fulbright Research Scholar. During his latest sabbatical he was a visiting professor at the Massachusetts Institute of Technology (MIT), Cambridge, and United Technologies Research Center (UTRC). He is coauthor with R. Tweedie of the monograph *Markov Chains and Stochastic Stability* (Springer-Verlag, 1993). His more recent book, *Control Techniques for Complex Networks*, is published by Cambridge University Press (2007). His research interests include stochastic processes, optimization, complex networks, and information theory, with applications to smarter cities and smarter grids.

Dr. Meyn received jointly with R. Tweedie the 1994 ORSA/TIMS Best Publication In Applied Probability Award.

Uday V. Shanbhag (M'09) received the B.S. degree from the Indian Institute of Technology (IIT), Bombay, in 1993, the M.S. degree from the Massachusetts Institute of Technology (MIT), Cambridge, in 1998, and the Ph.D. degree in operations research from the Department of Management Science and Engineering, Stanford University, Stanford, CA, in 2006.

He is currently an Assistant Professor in industrial and enterprise systems engineering at the University of Illinois at Urbana-Champaign and his interests lies in the development of analytical and algorithmic tools in the context of optimization and game-theoretic problems, in regimes complicated by uncertainty, dynamics, and nonsmoothness.

Dr. Shanbhag received the triennial A. W. Tucker Prize for his dissertation from the mathematical programming society (MPS) in 2006 and the Computational Optimization and Applications (COAP) Best Paper Award in 2007.